Subgraph characterization of red-blue split graphs and König-Egerváry graphs

Thanh Nguyen

Joint work with Ephraim Korach and Britta Peis. (SODA 06)

History

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What are the graphs satisfying: min vertex cover = max matching ?

- Semi bipartite graphs,
- Graphs with König property,
- König-Egerváry Graphs (KEGs) .

Examples:

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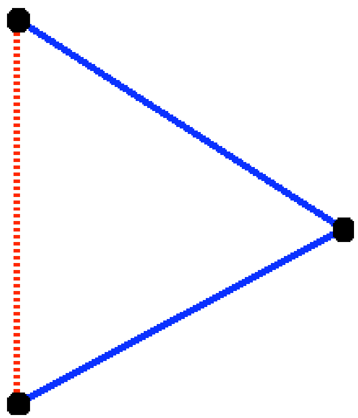

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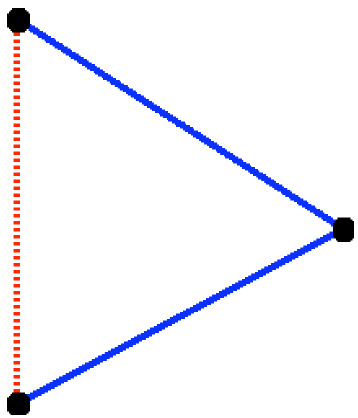

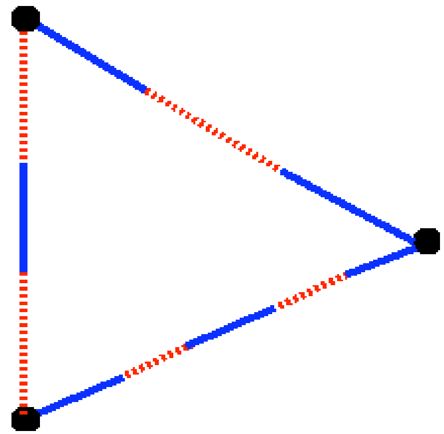

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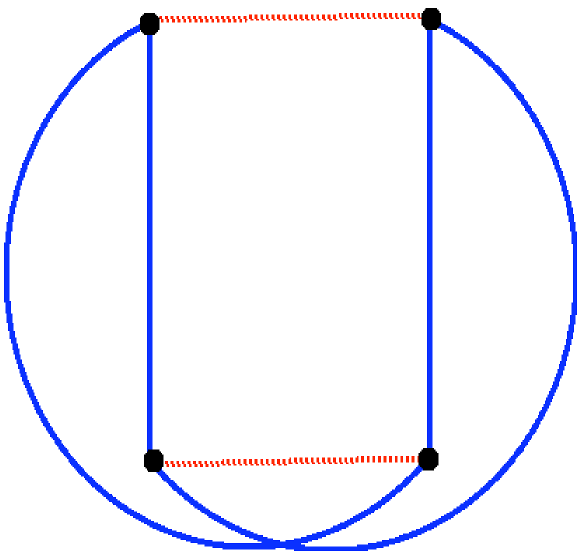

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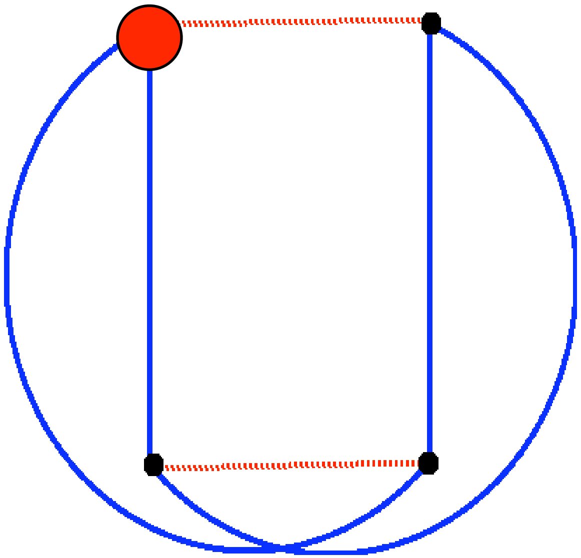

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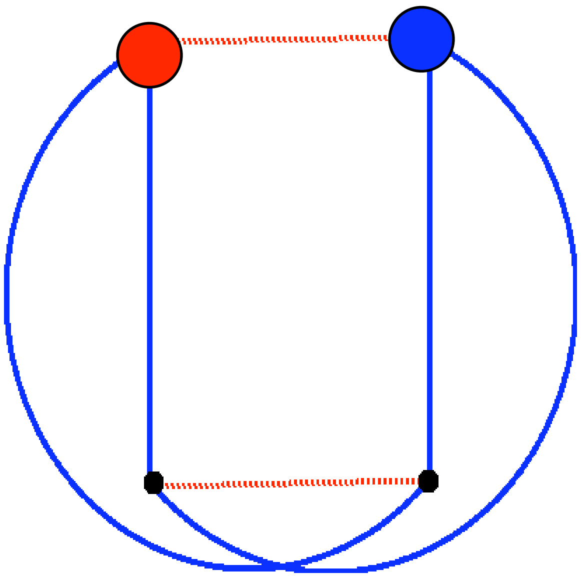

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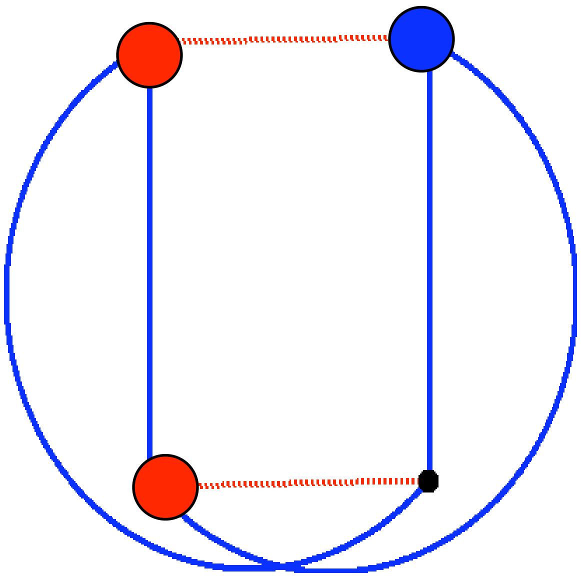

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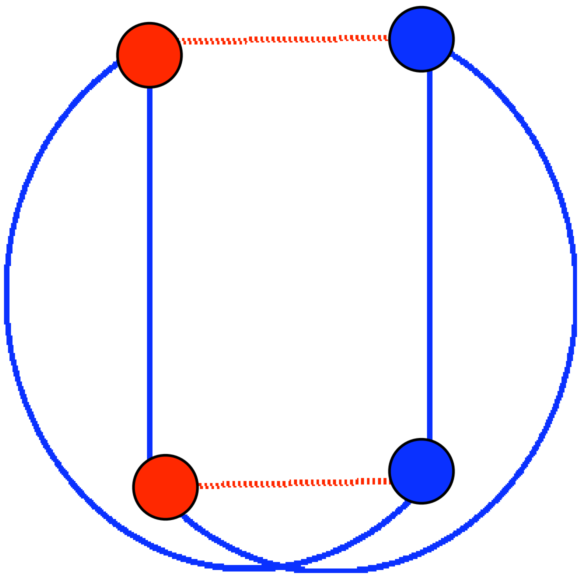

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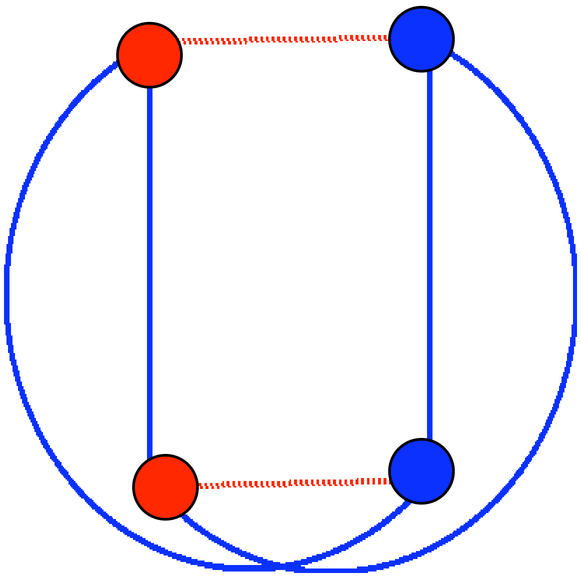

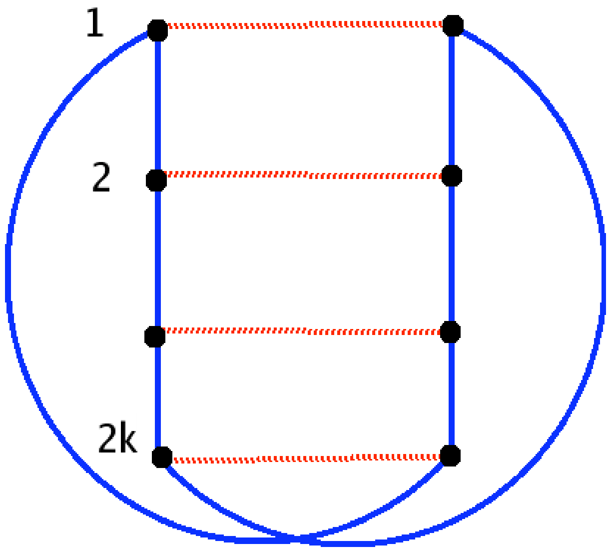

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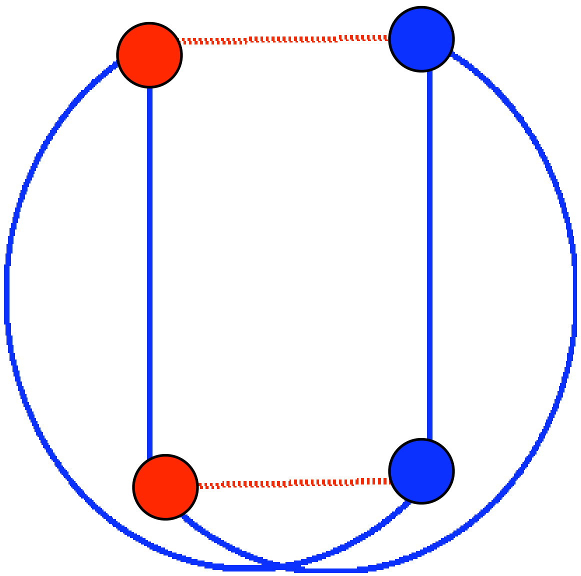
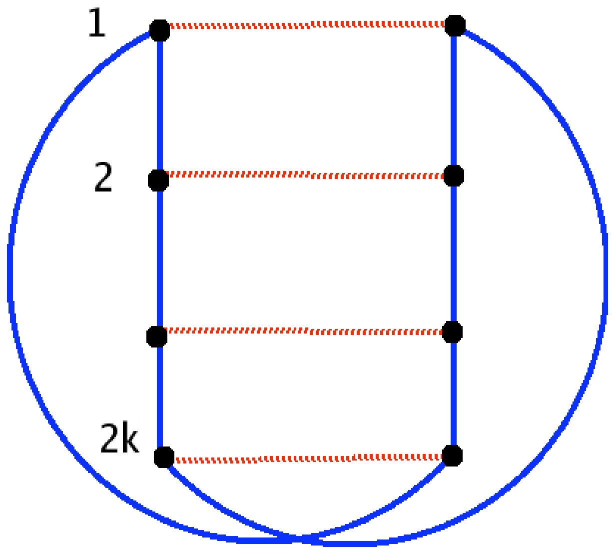




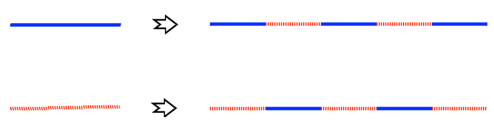
❖Works if we replace every edge by an odd length path.

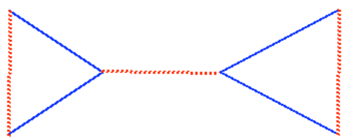
Main theorem

❖ Theorem: Given a graph and a maximum matching, it is KEG iff doesn't contain one of the following subgraphs.

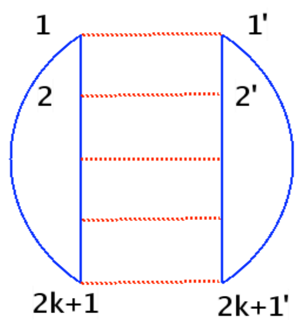
❖Each edge is a path

❖v is not covered by matching edges.

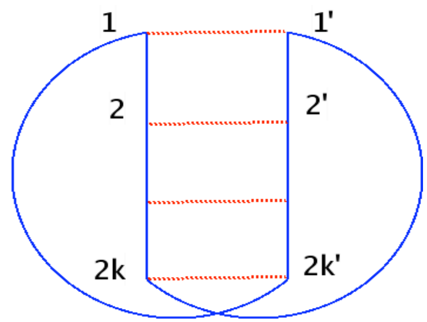




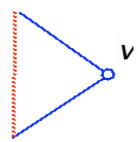
Triangular blossom pair



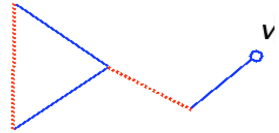
Odd prism



Even Mobius prism



Triangular



Triangular flower

Why subgraph characterization?

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- The traditional example is the case of planar graphs
  - Hopcroft algorithm
  - Kuratowski theorem: K5, K3,3

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- 1977 Sterboul, Deming：algorithm
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  - less general
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- Our result:
  - A stronger characterization
  - A more general model (red/blue split graphs)

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- A general model captures both split graphs and KEGs:
  - red/blue split graph: graph whose edges are colored red and blue
  - can be partitioned into a red and a blue independent sets.

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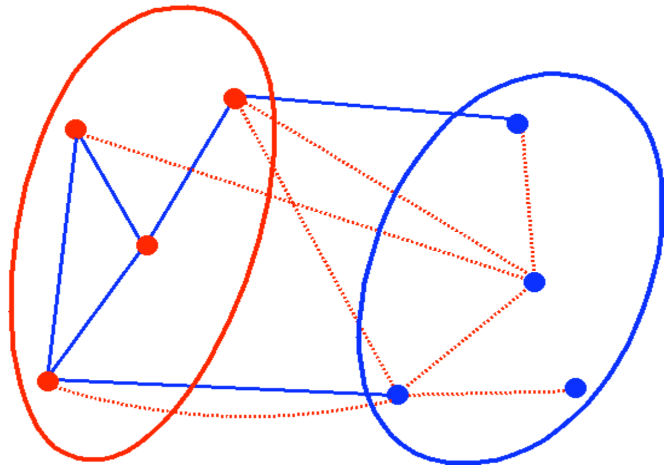

Red blue split graphs and KEGs

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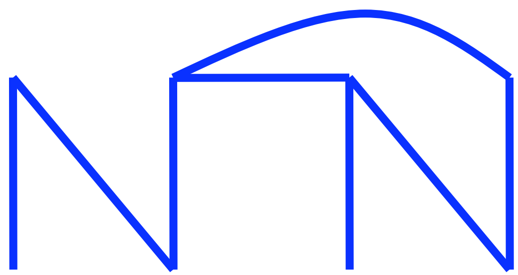

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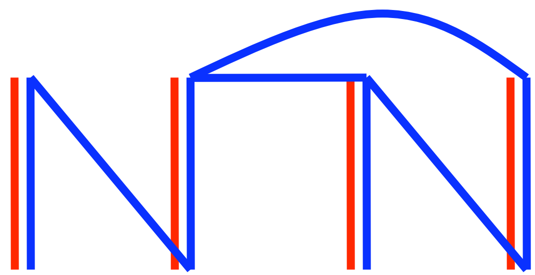

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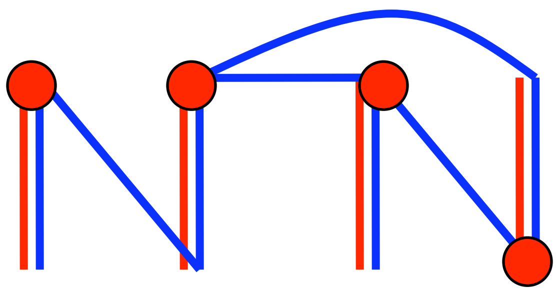

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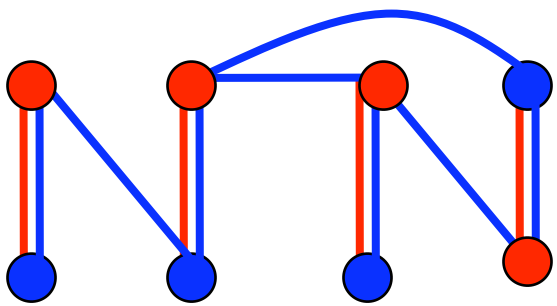

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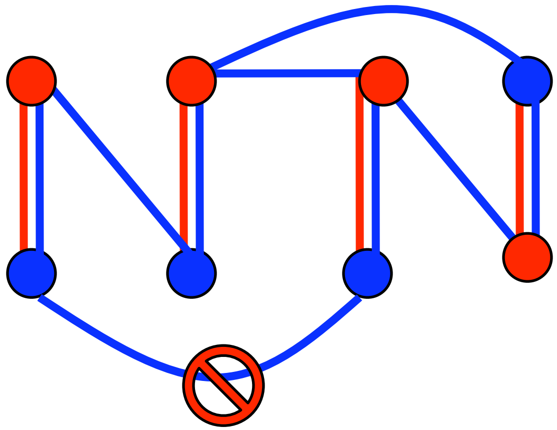

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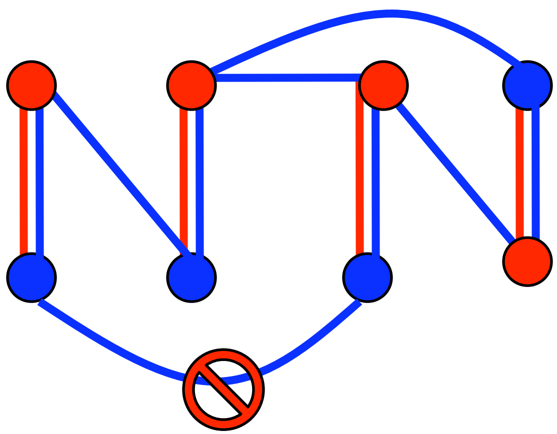

- Only works for graph with perfect matching !!!
- General case later.

To recognize red blue split graphs

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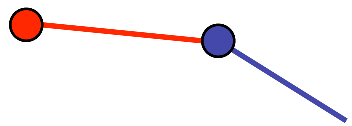

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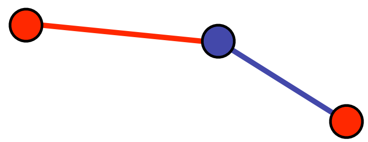

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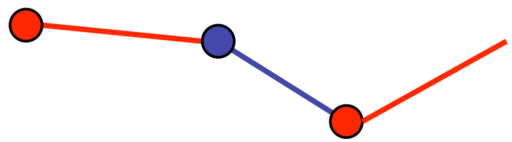

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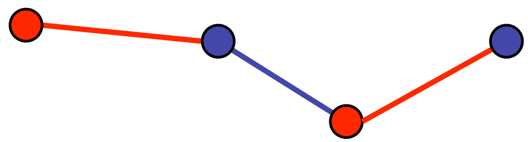

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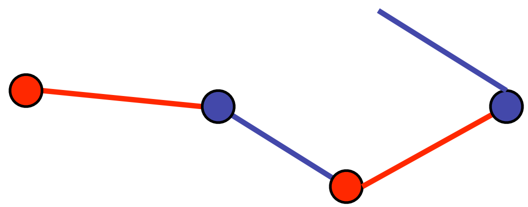

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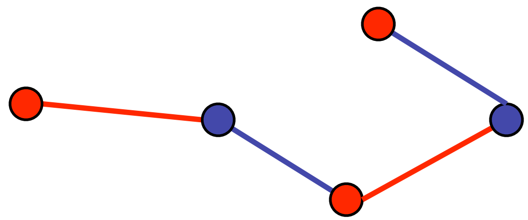

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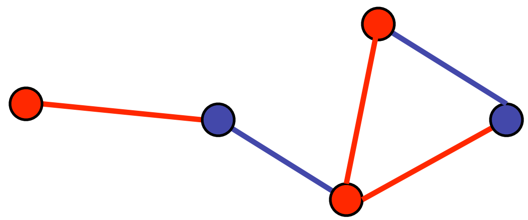

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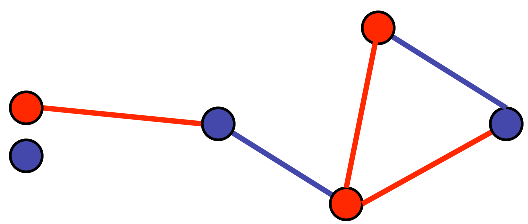

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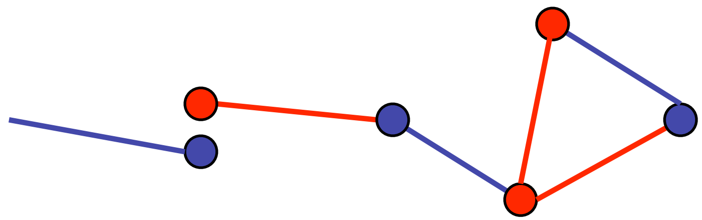

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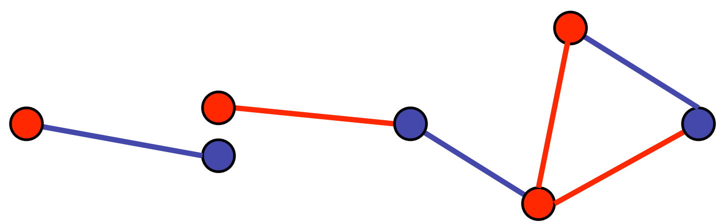

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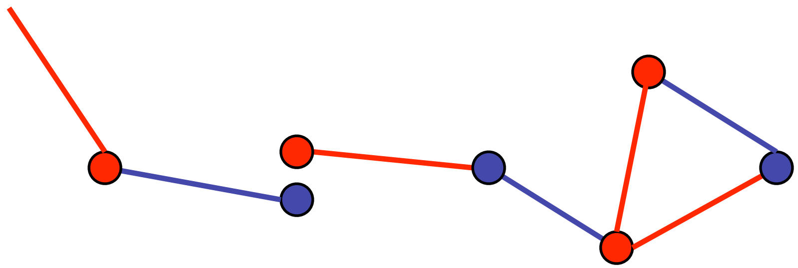

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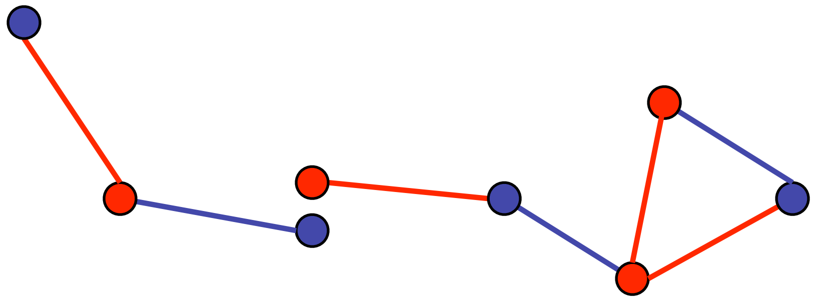

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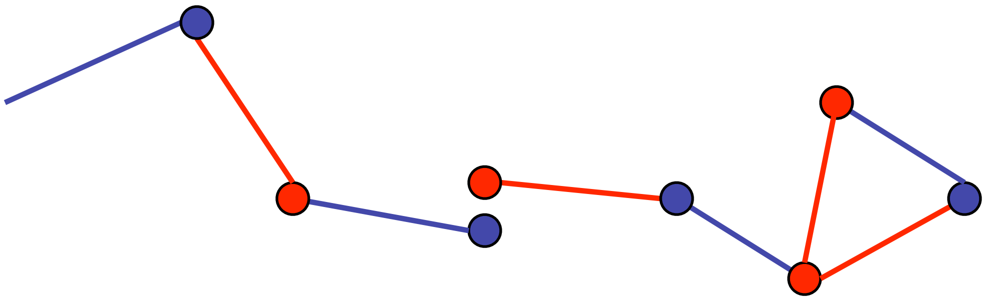

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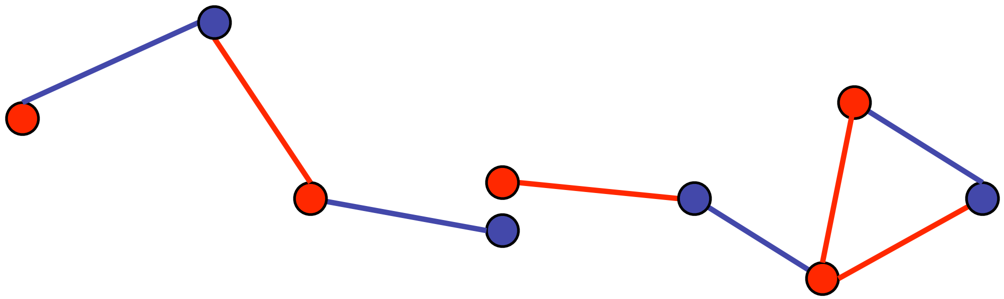

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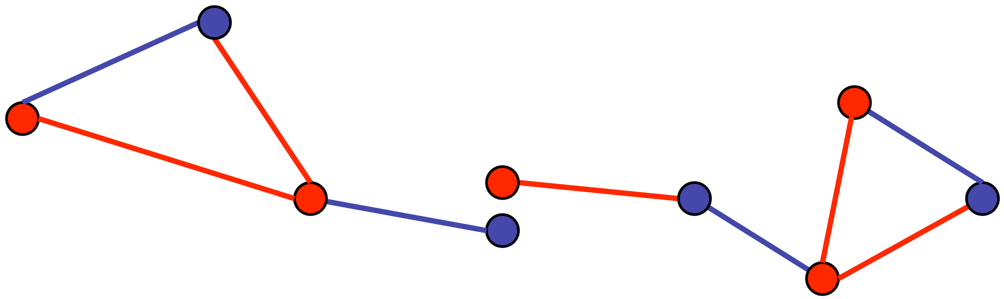

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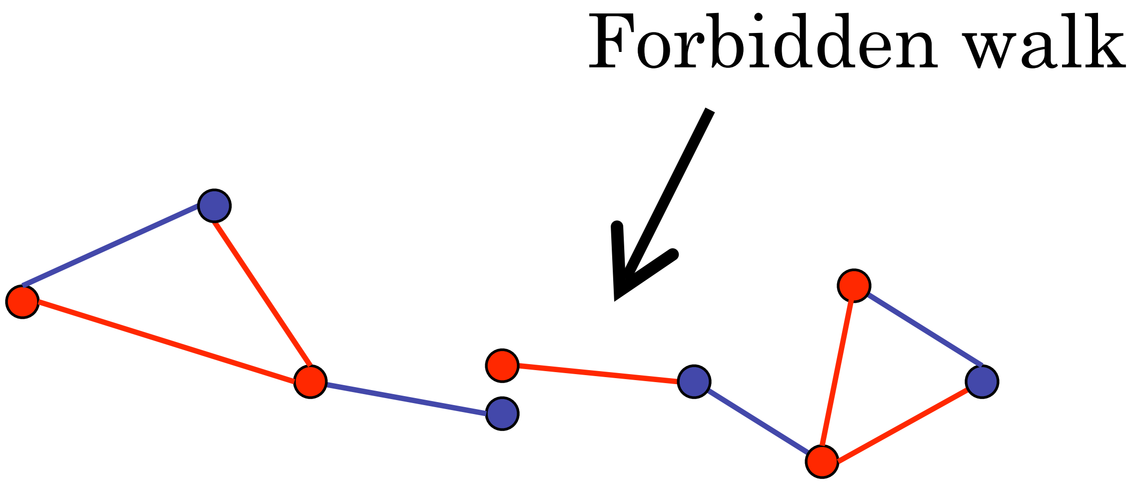
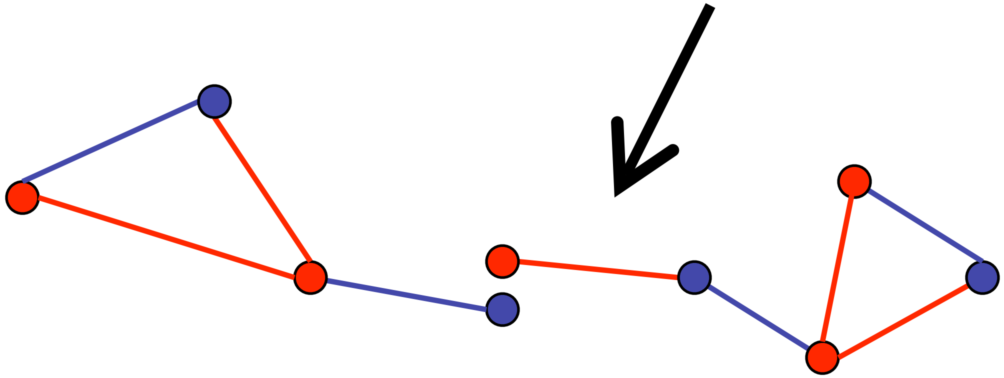


To recognize red blue split graphs

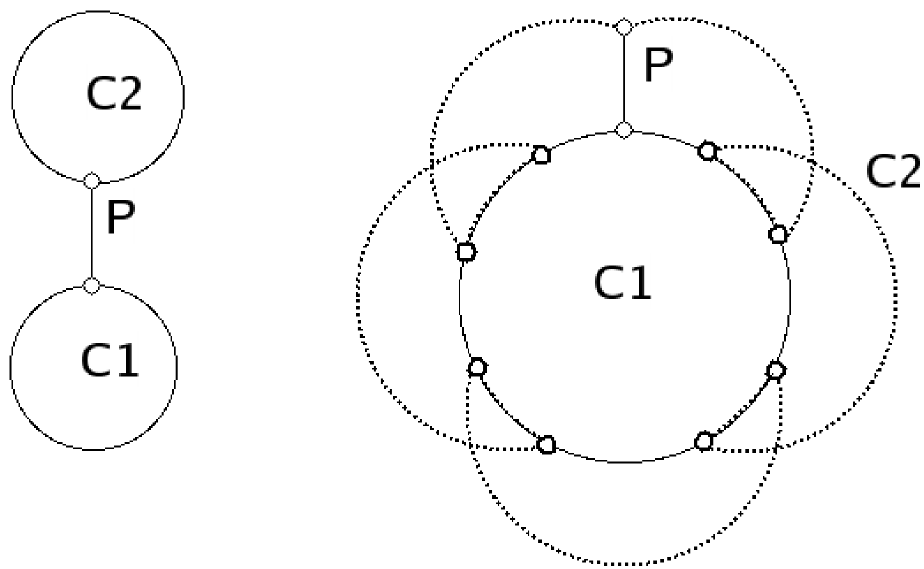
Forbidden walk



Not a subgraph!!!

Subgraph characterization of red blue split graphs

❖Theorem: A graph is RB split iff it doesn't contain the following "bad" subgraphs.



RB split

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Forbidden walks

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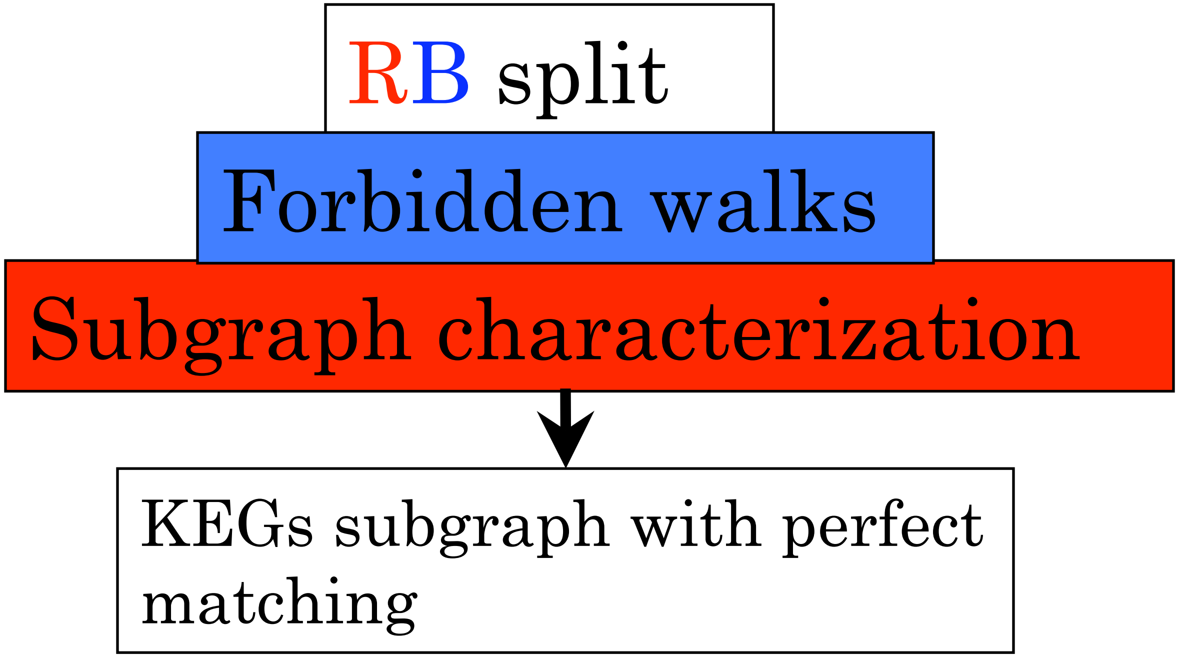
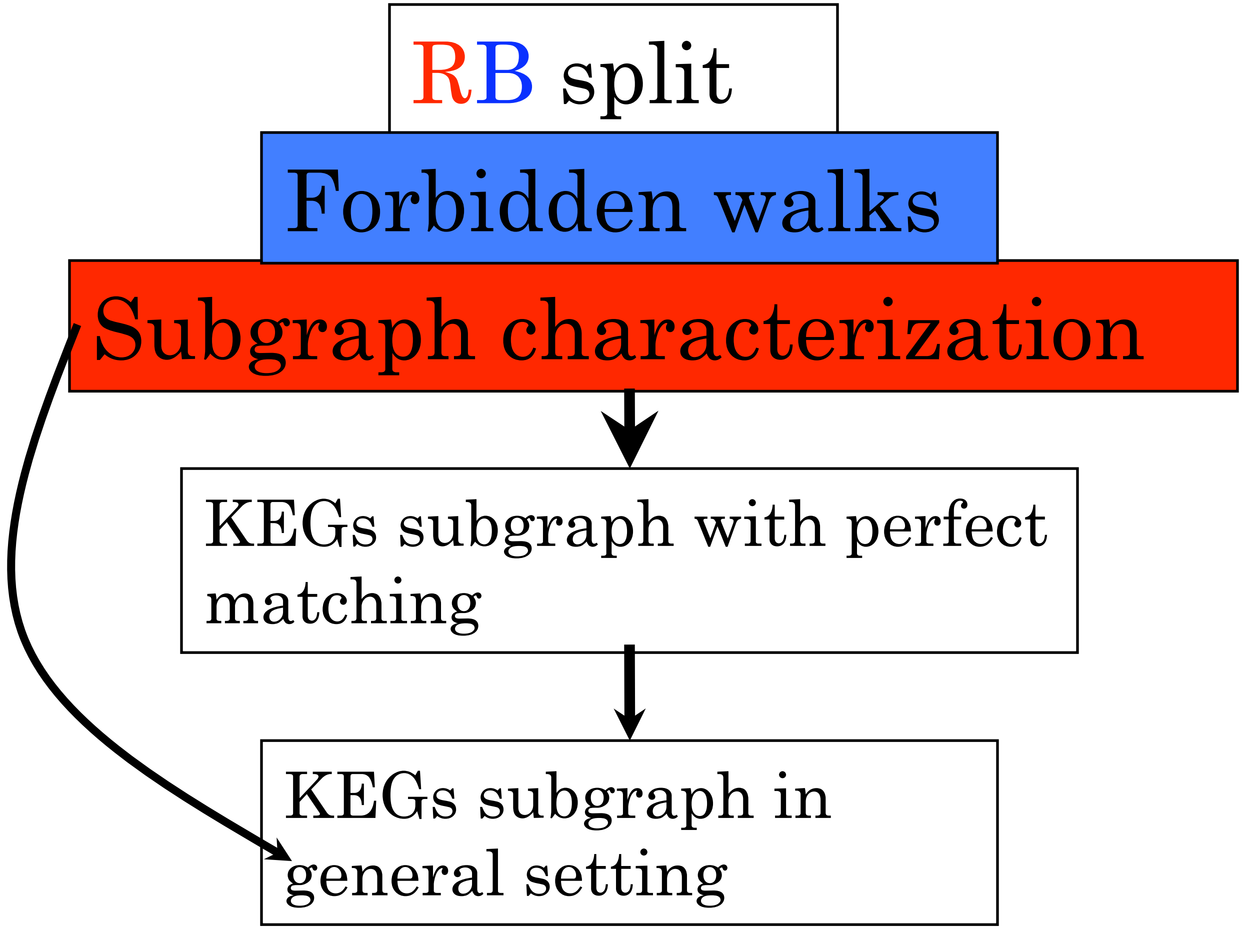
Subgraph characterization


RB split
Forbidden walks
Subgraph characterization
KEGs subgraph with perfect matching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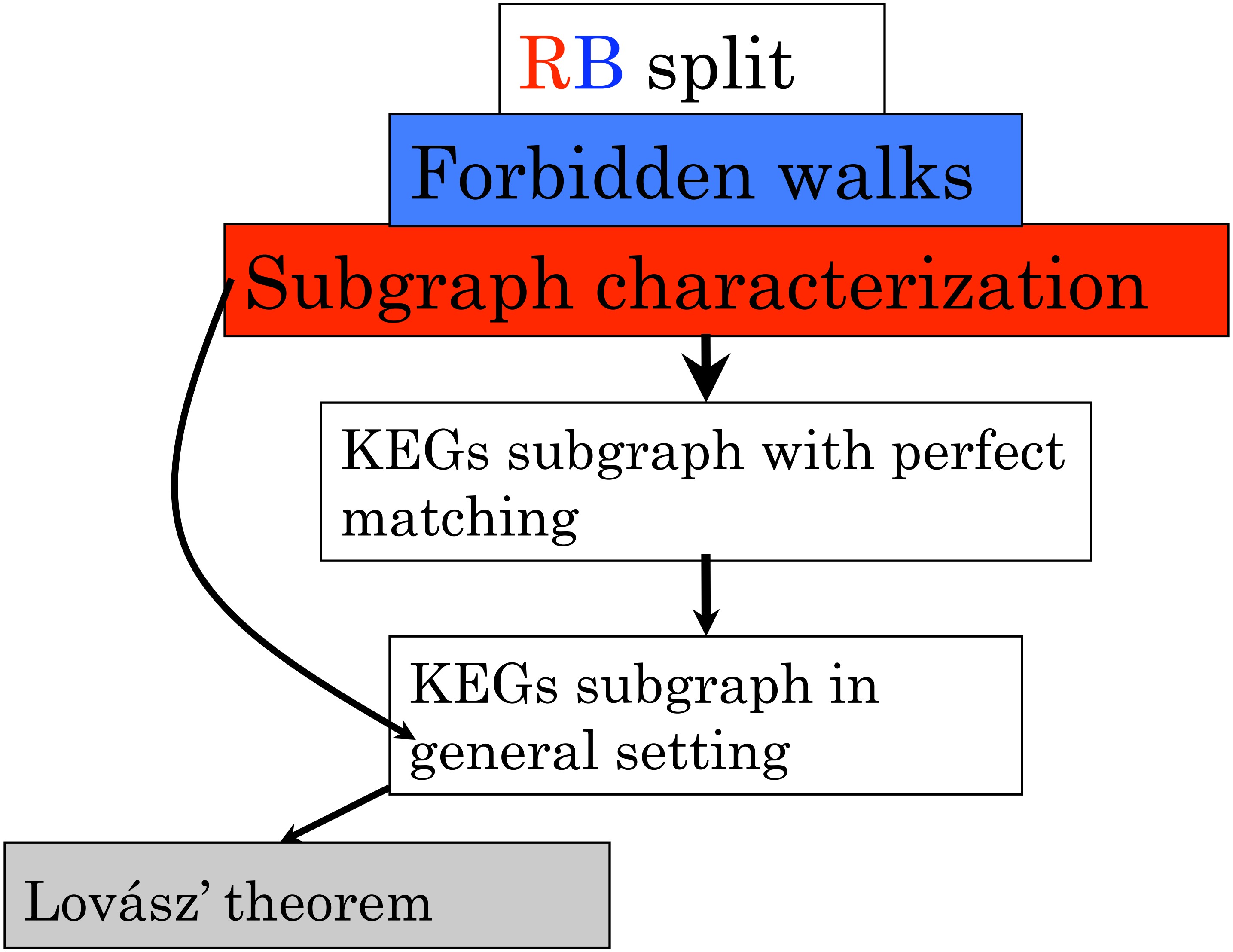
KEGs subgraph in general setting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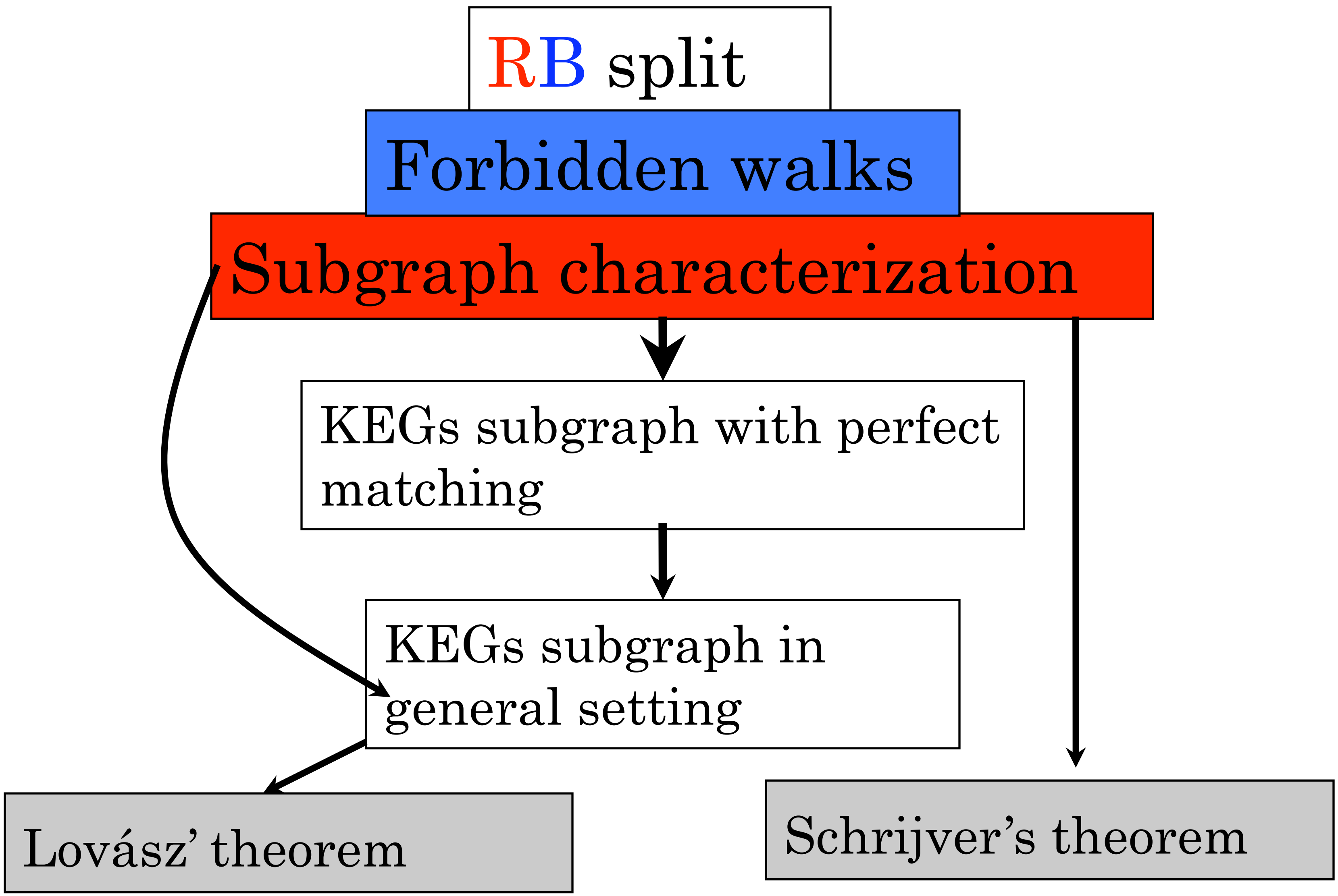
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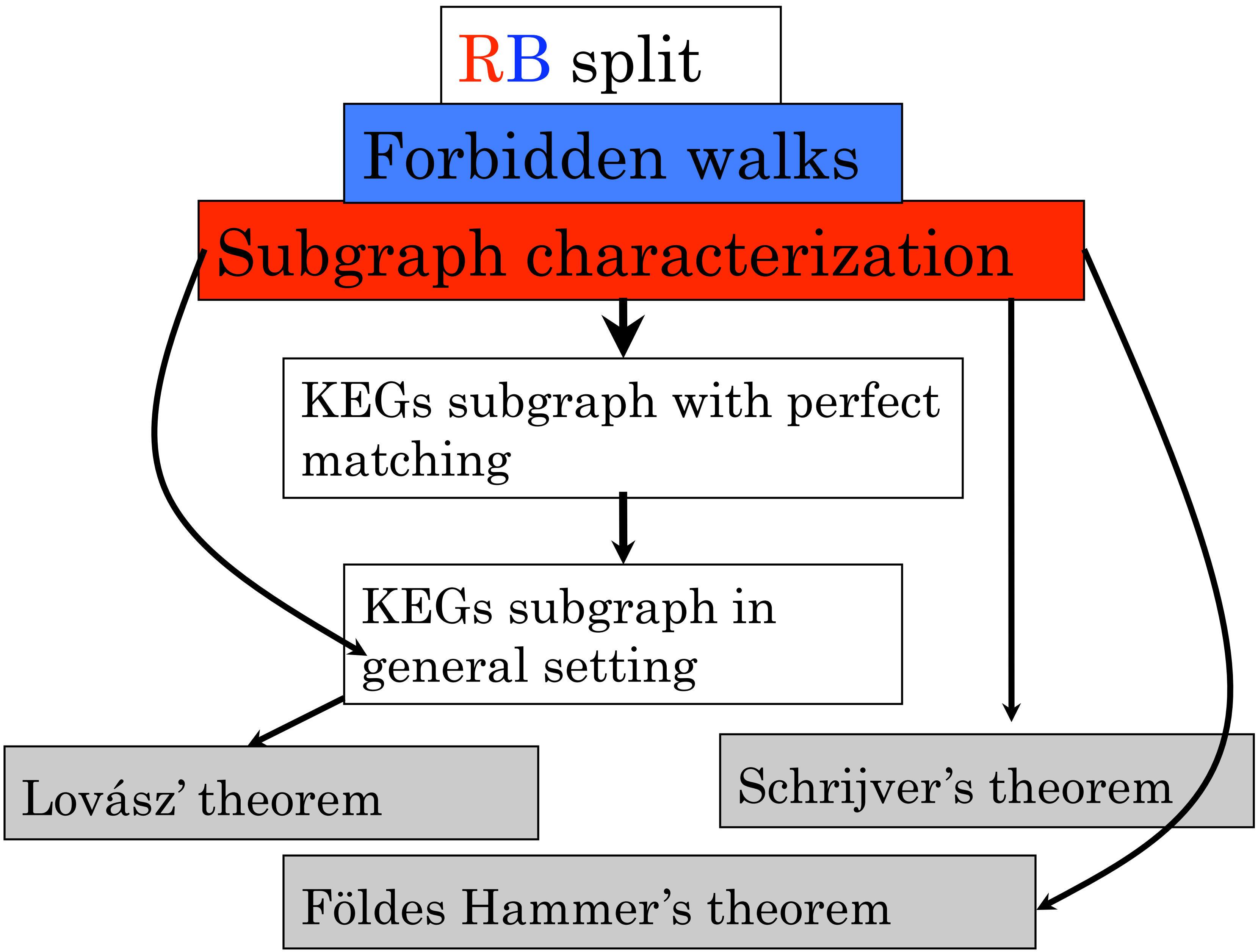
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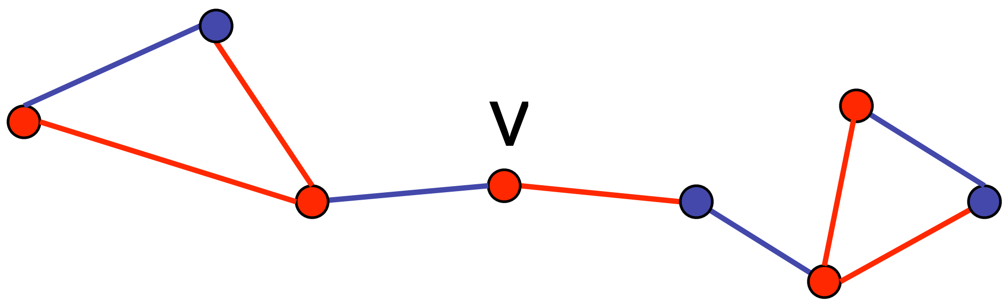
Forbidden cycles

Forbidden cycles



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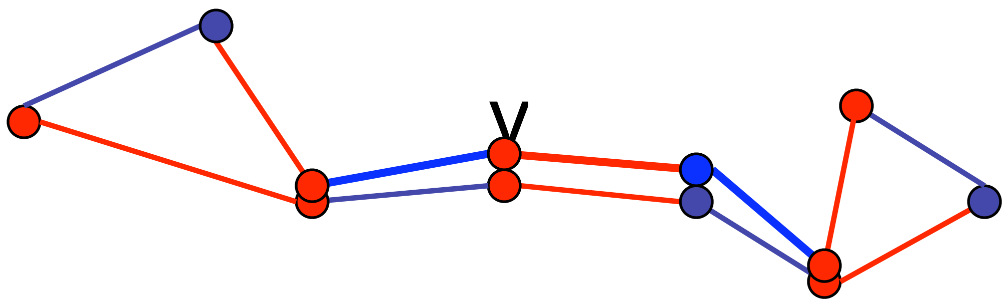

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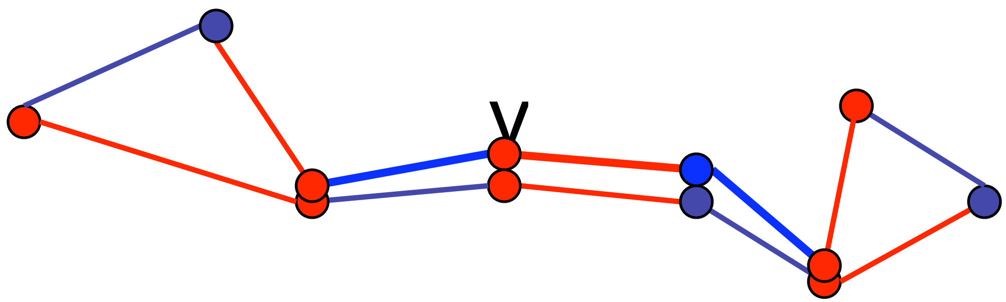

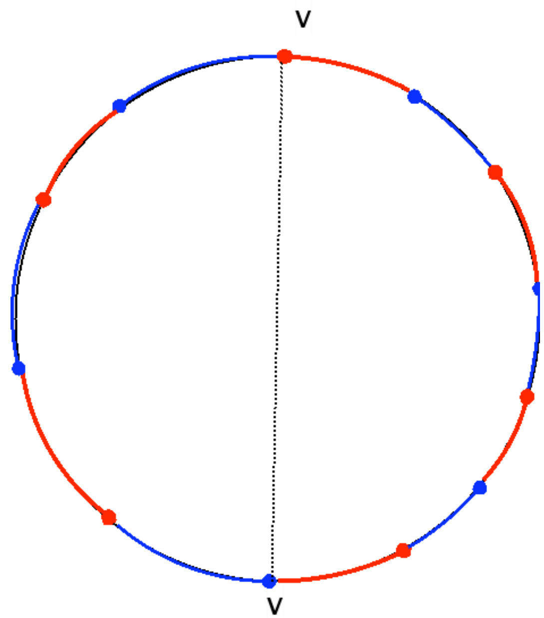

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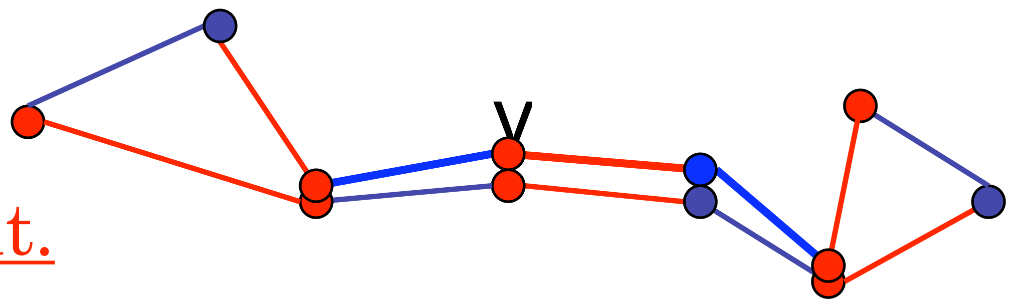
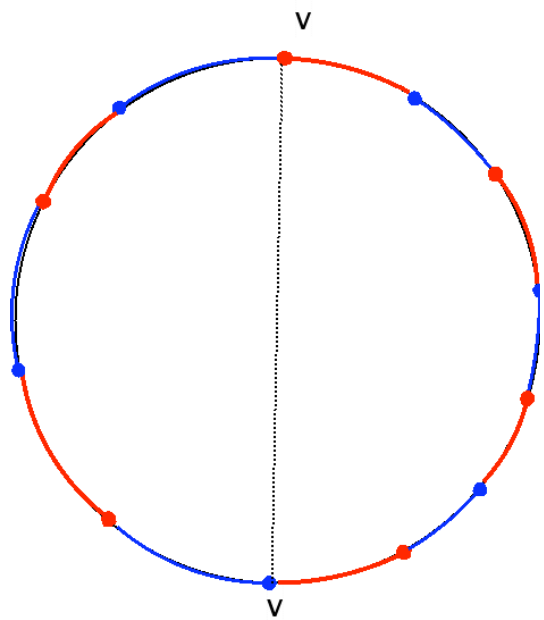


- ❖ Odd cut.



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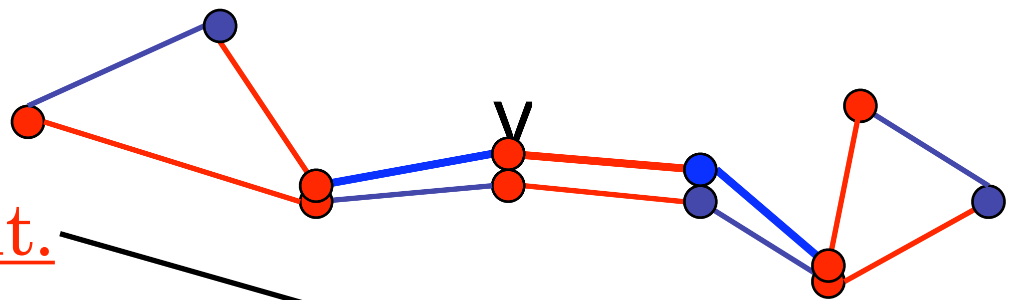

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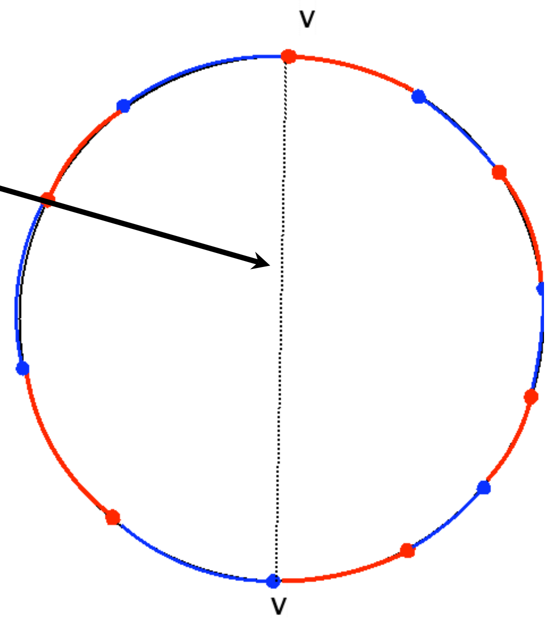

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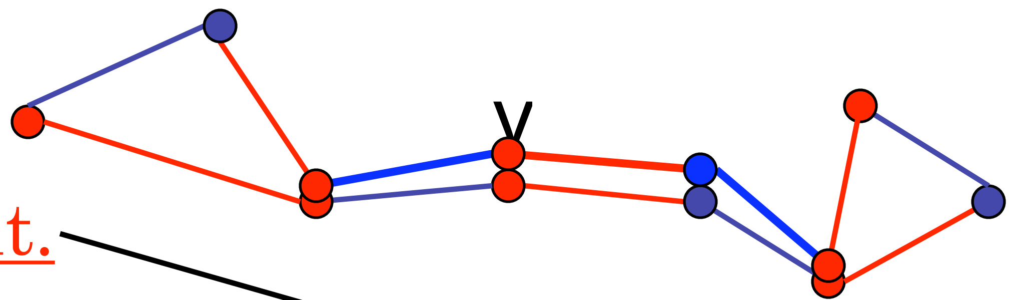

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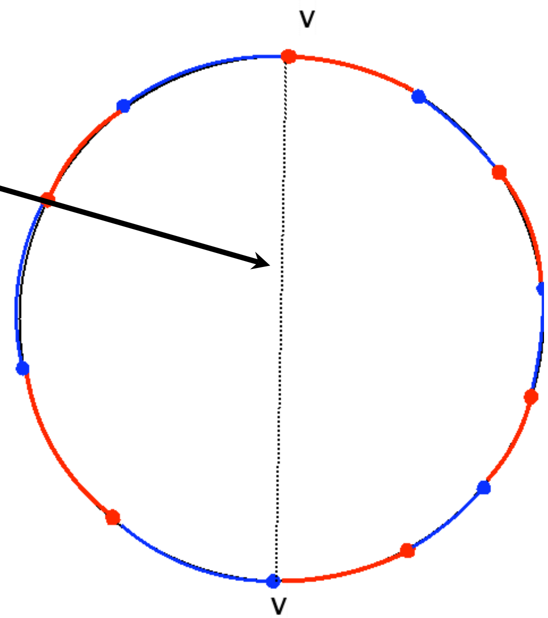
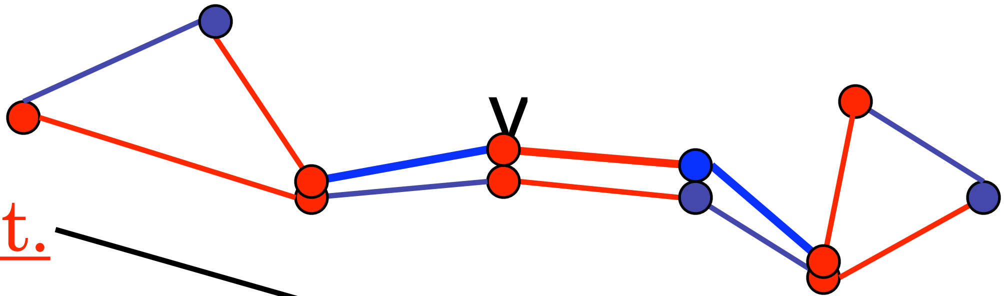


- Forbidden cycle = alternating cycle with an odd cut.

Forbidden cycles



❖Odd cut.

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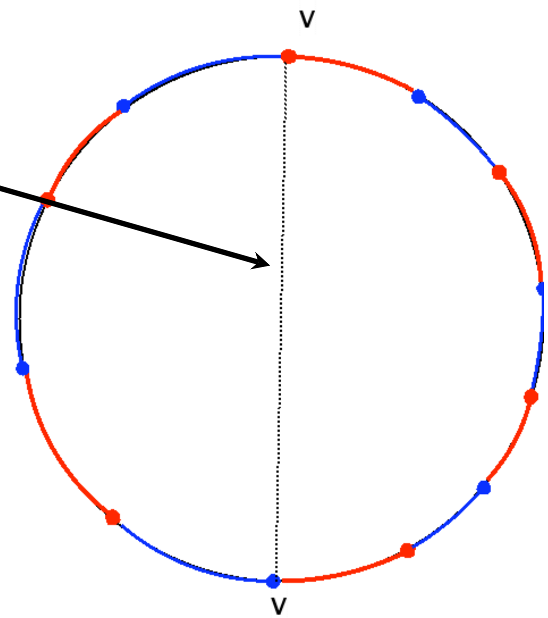

✓Theorem: RB split iff NO forbidden cycles.

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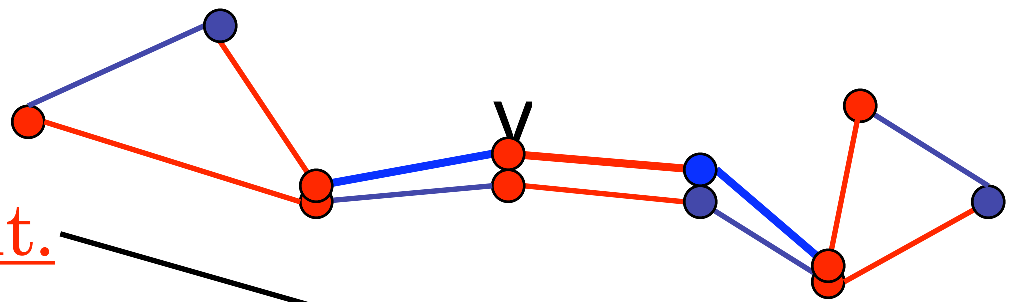

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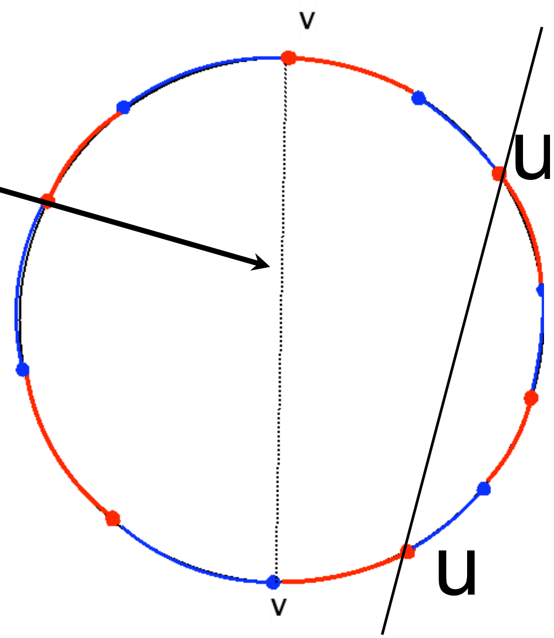

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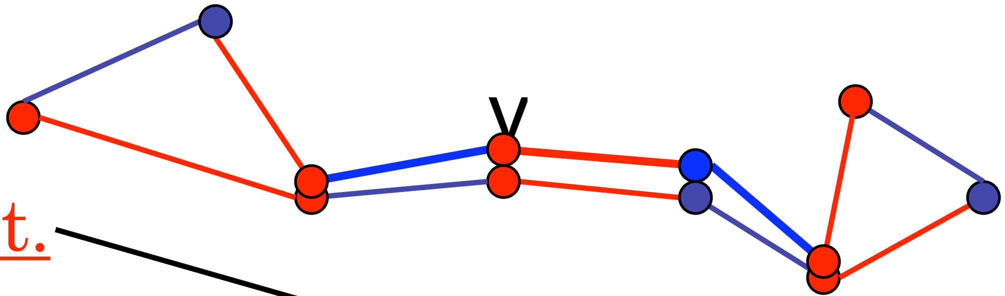

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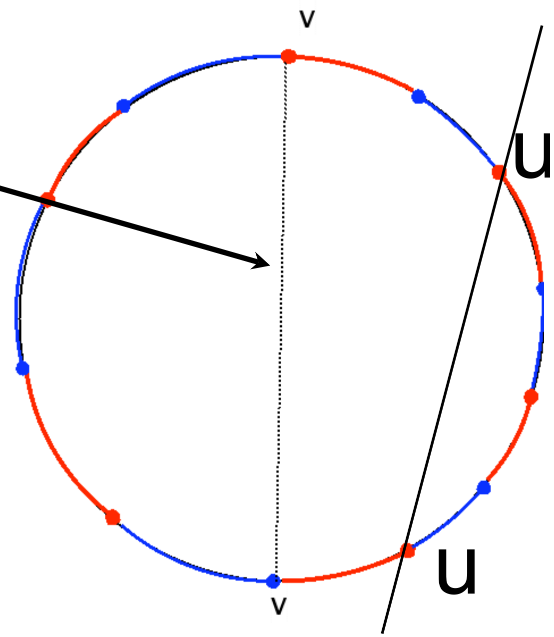

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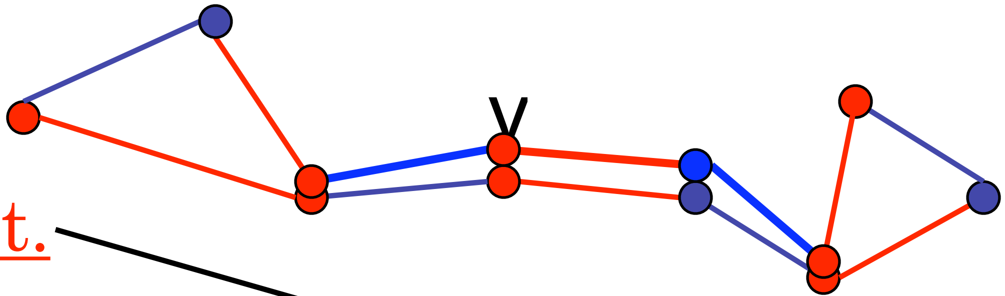

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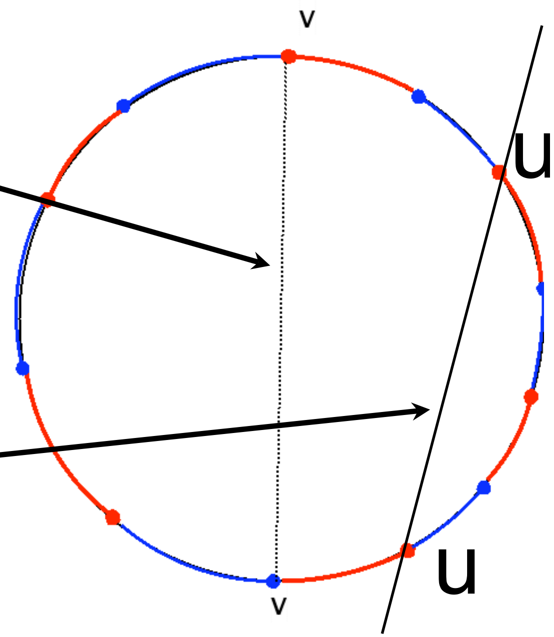

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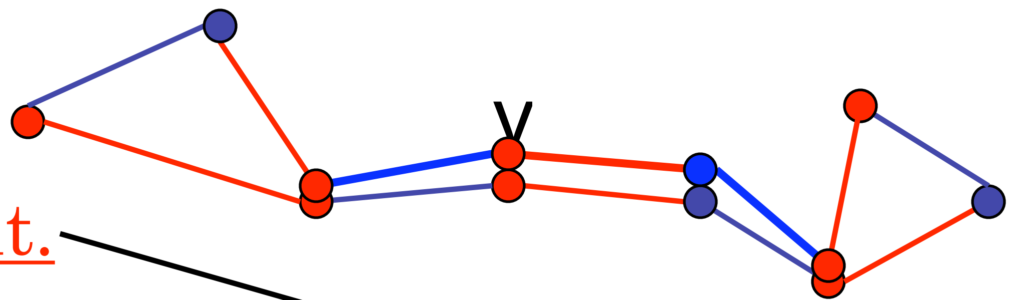

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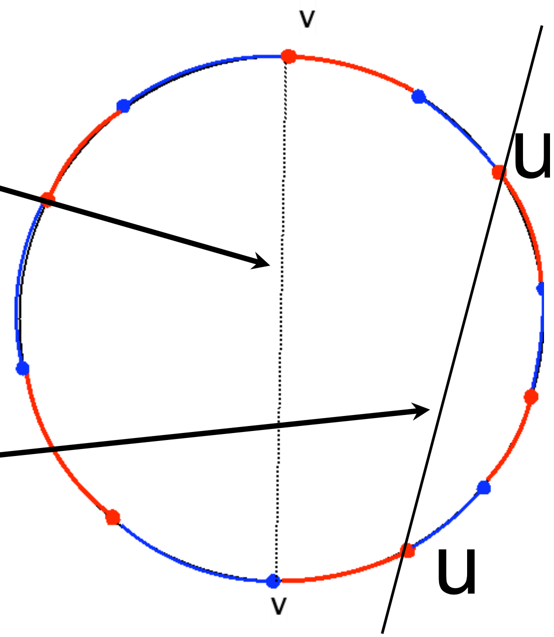
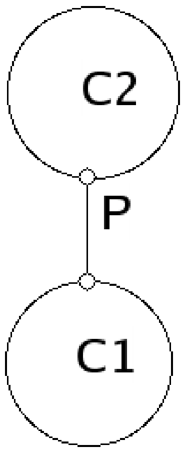


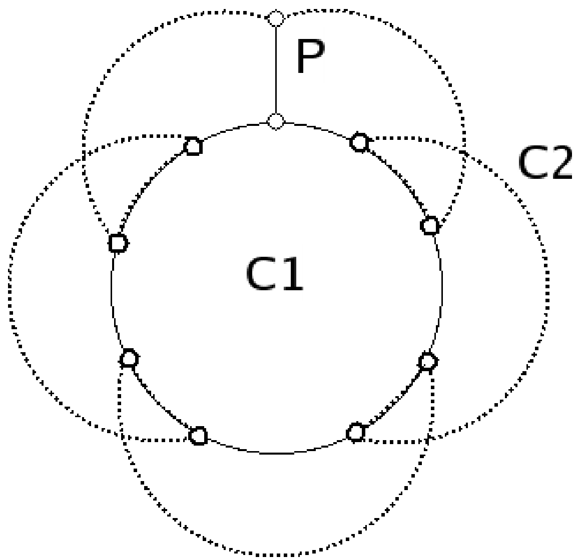
❖Even cut

❖Parallel cuts, crossing cuts

Subgraph characterization of red blue split graphs

- Goal: to turn a forbidden cycle into one of the forbidden subgraphs!





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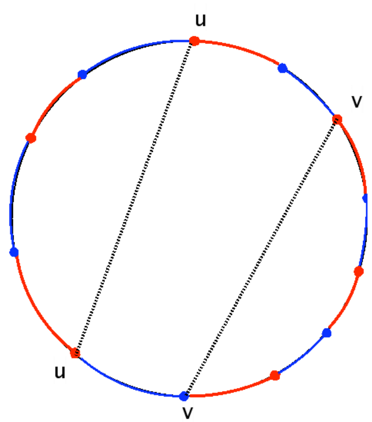

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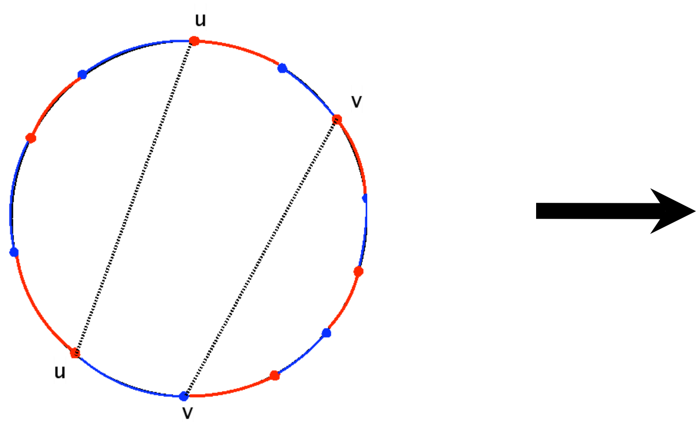

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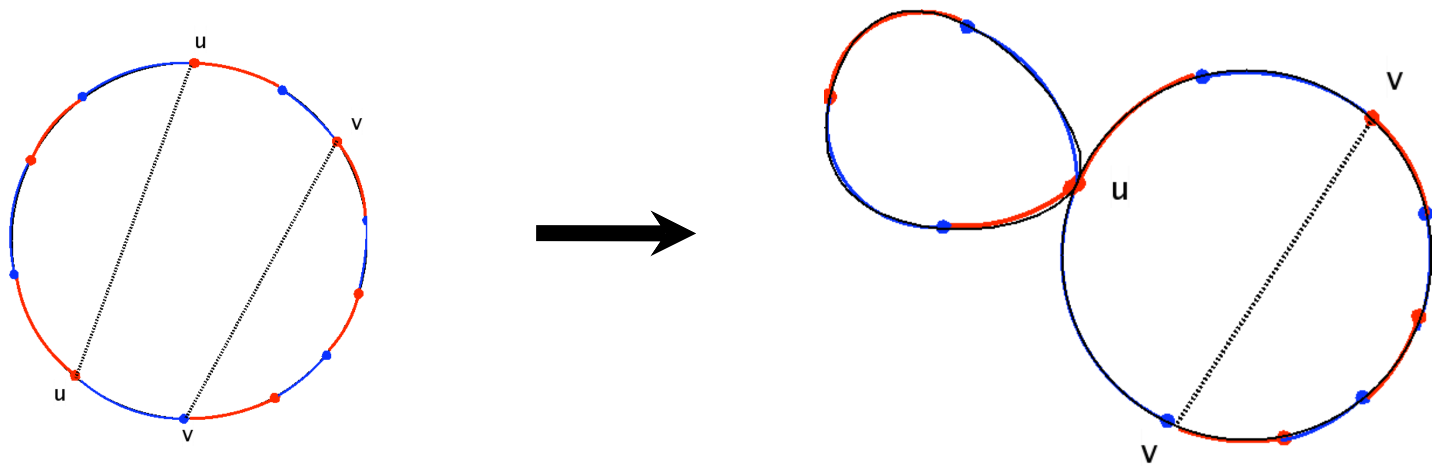

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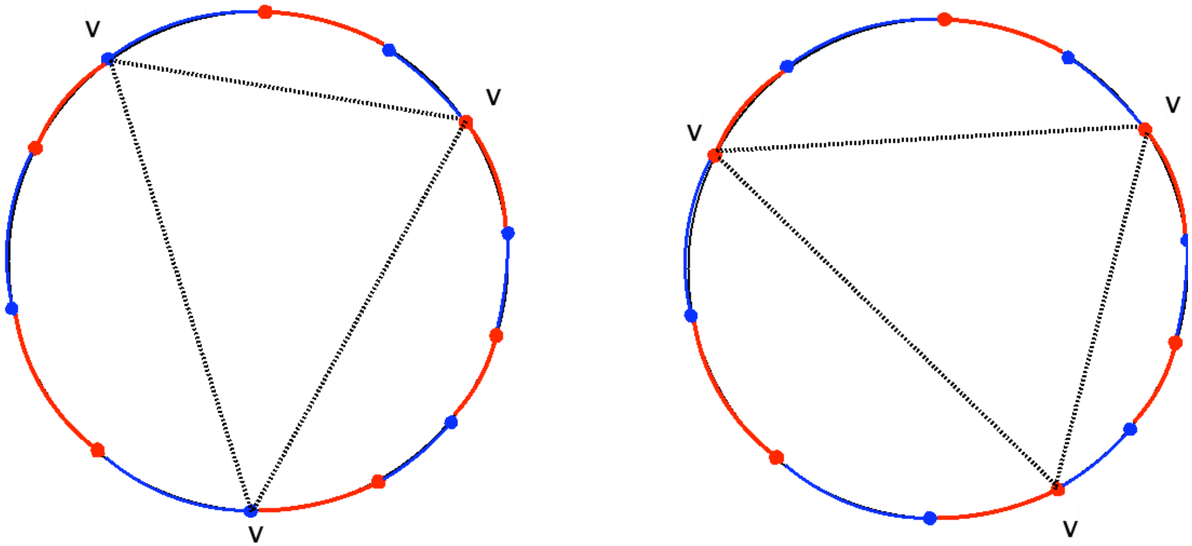

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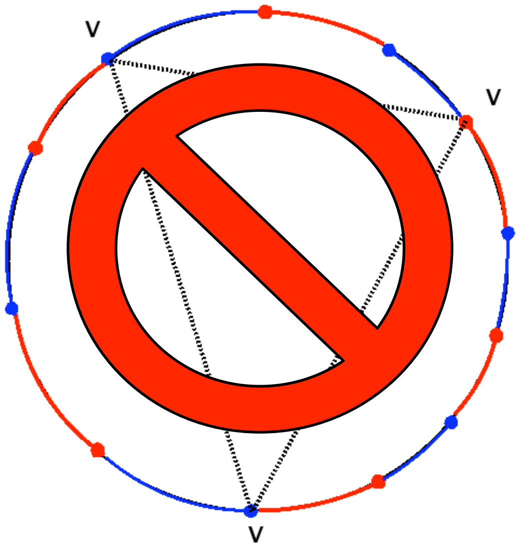

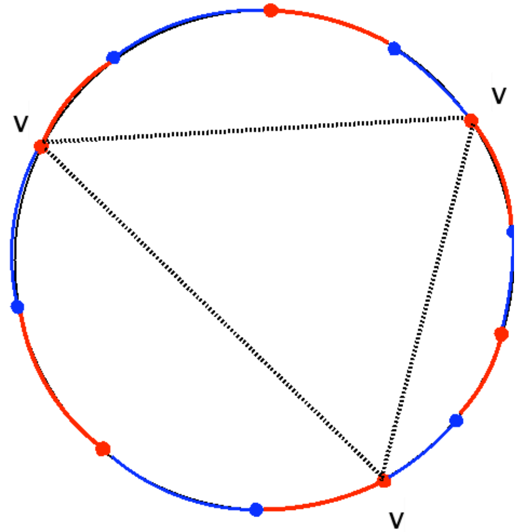

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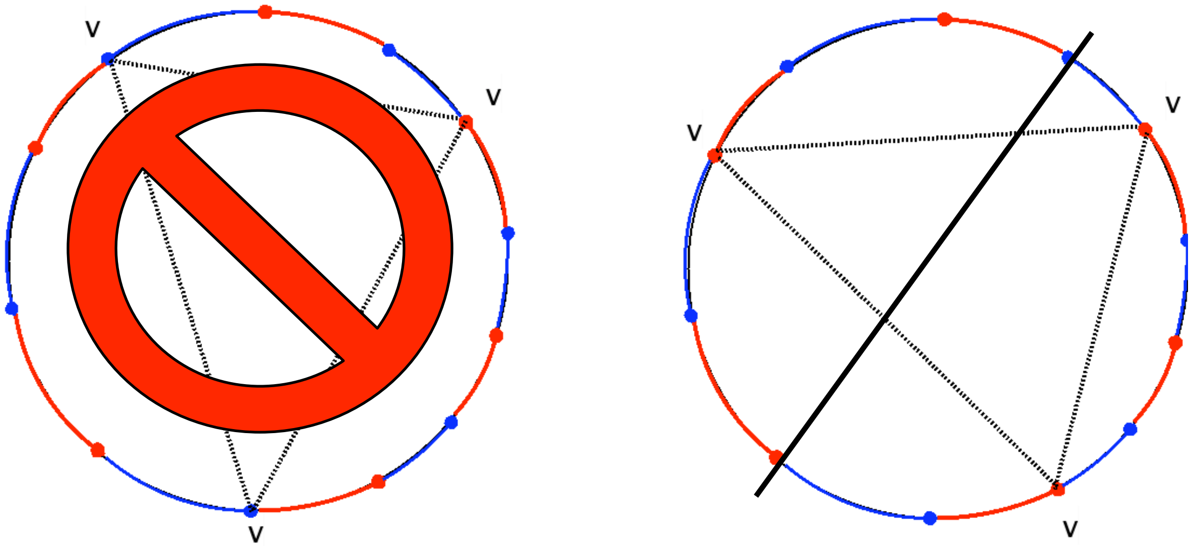

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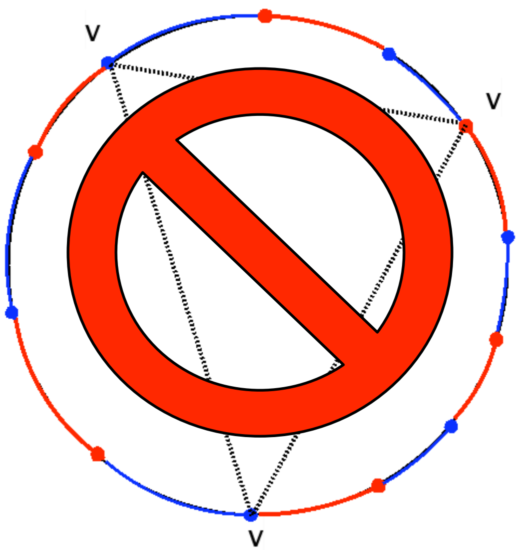

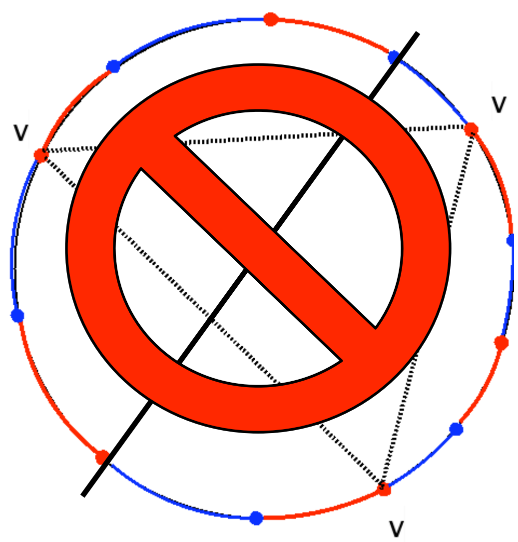

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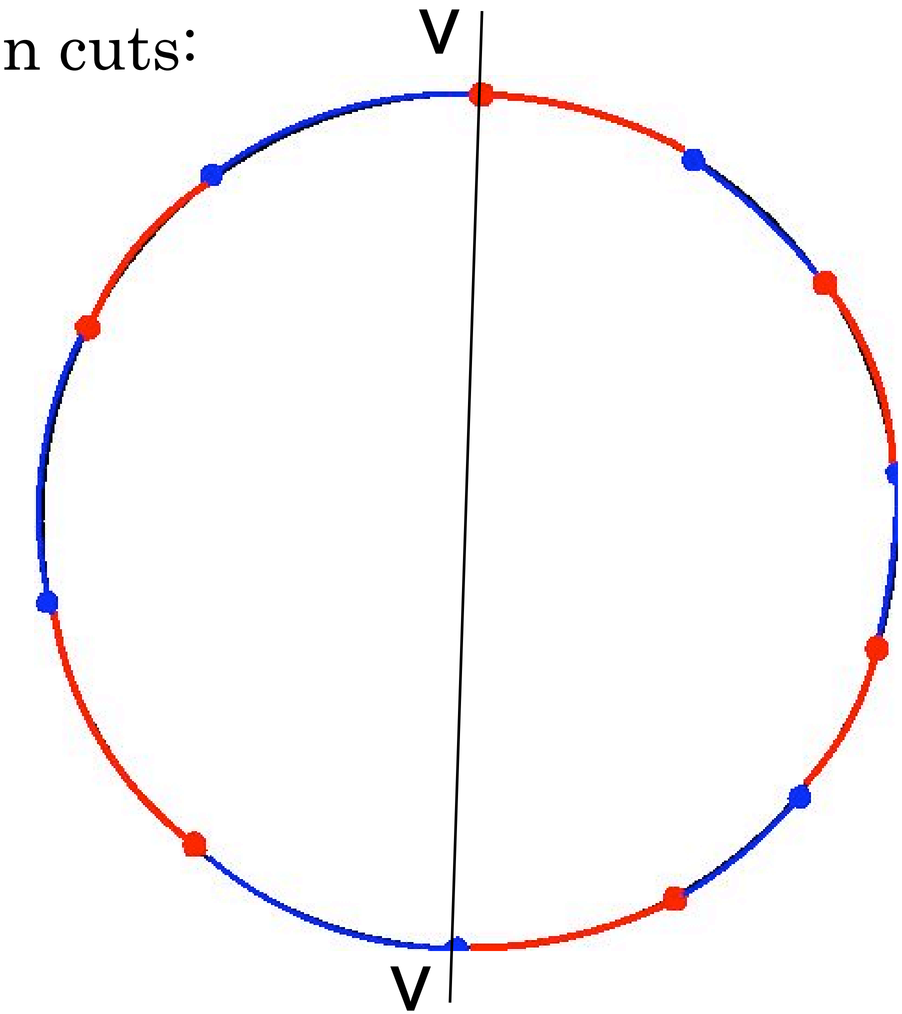

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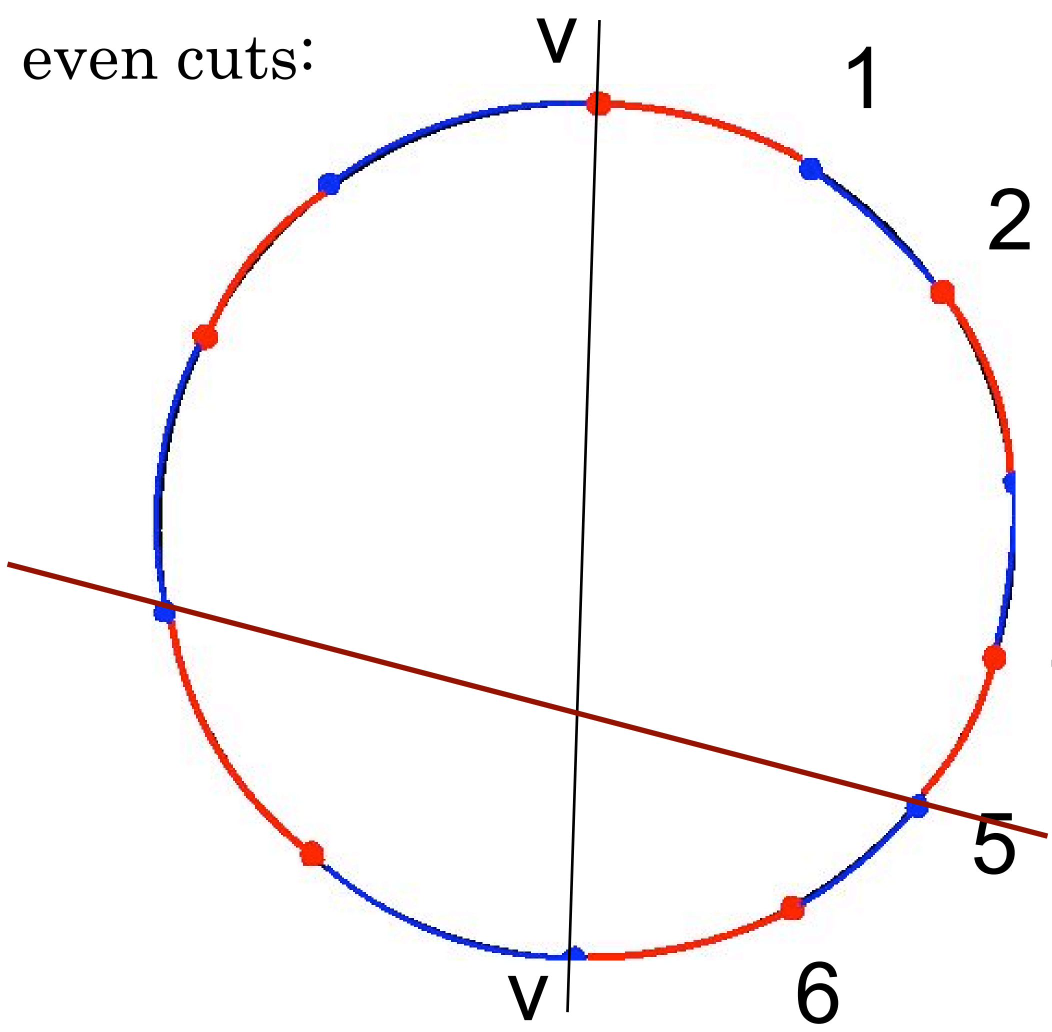

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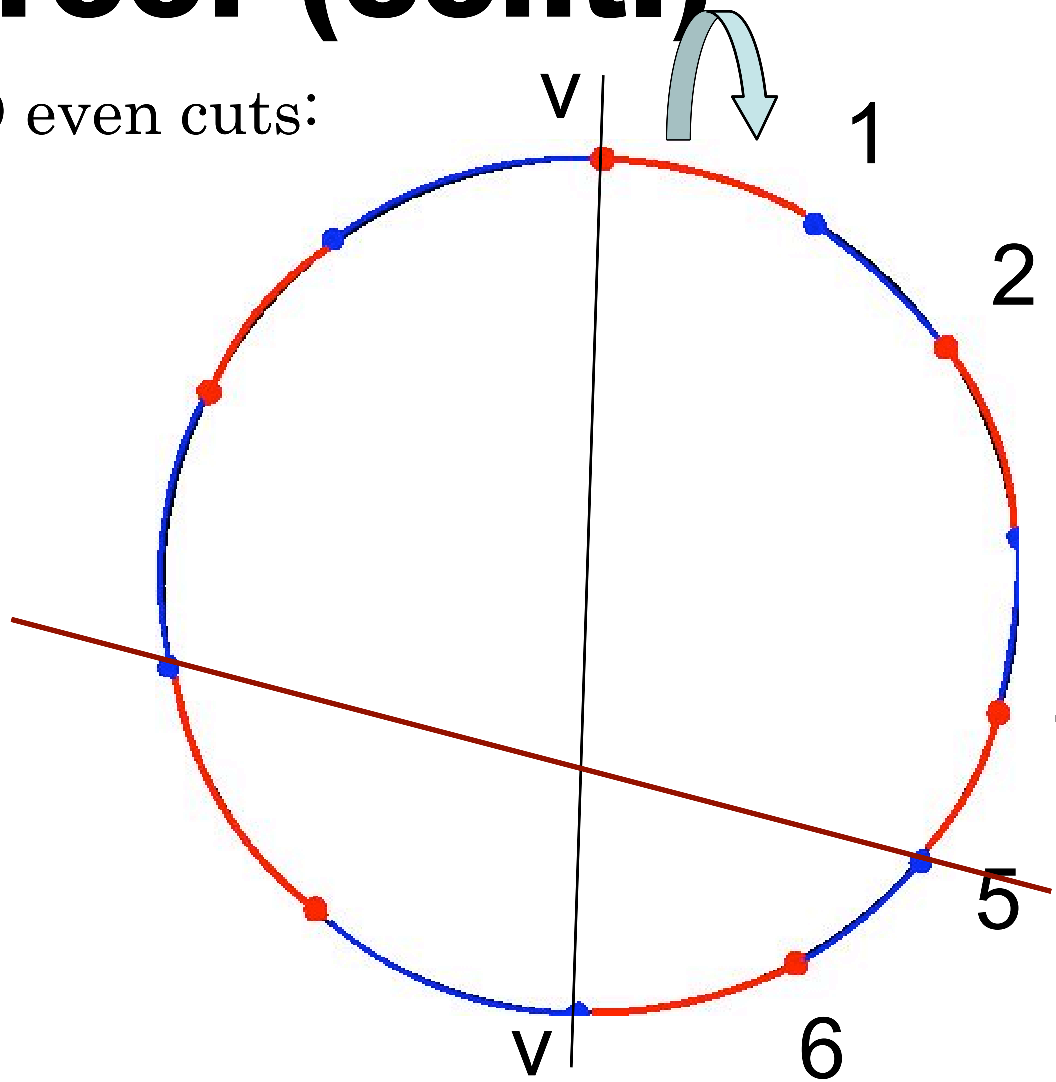

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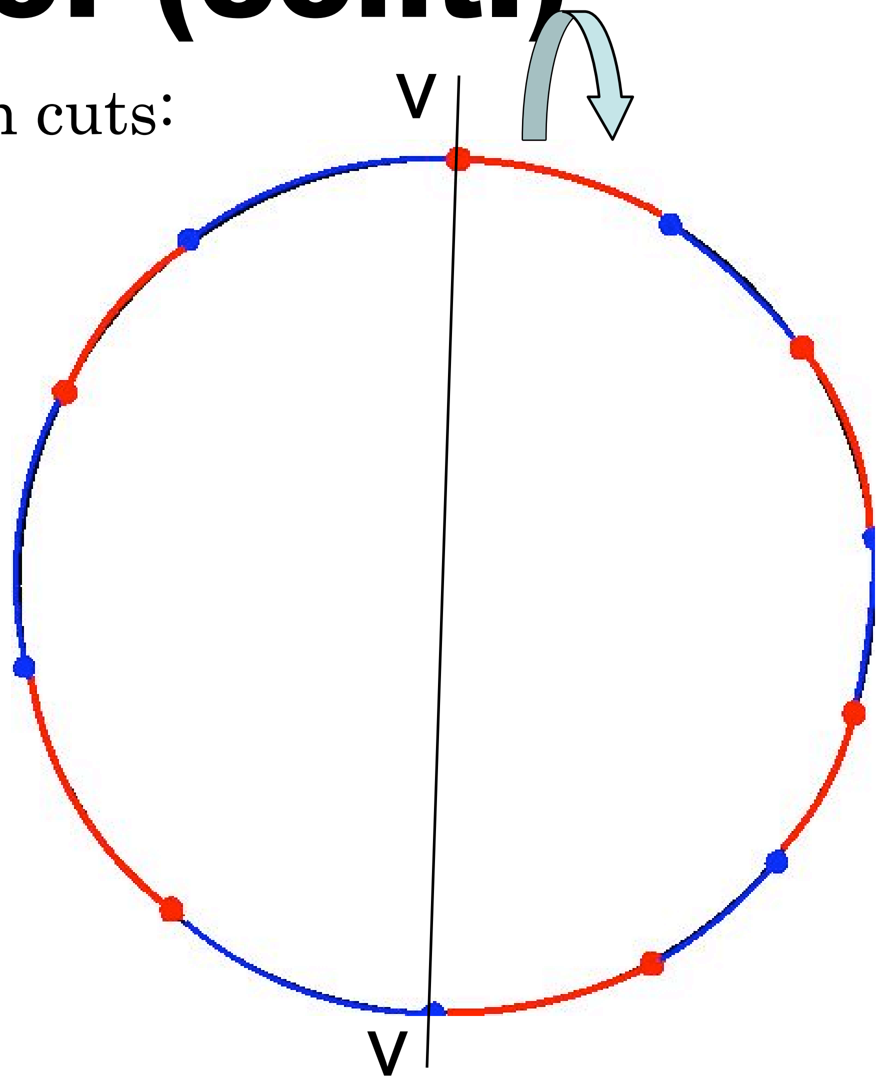

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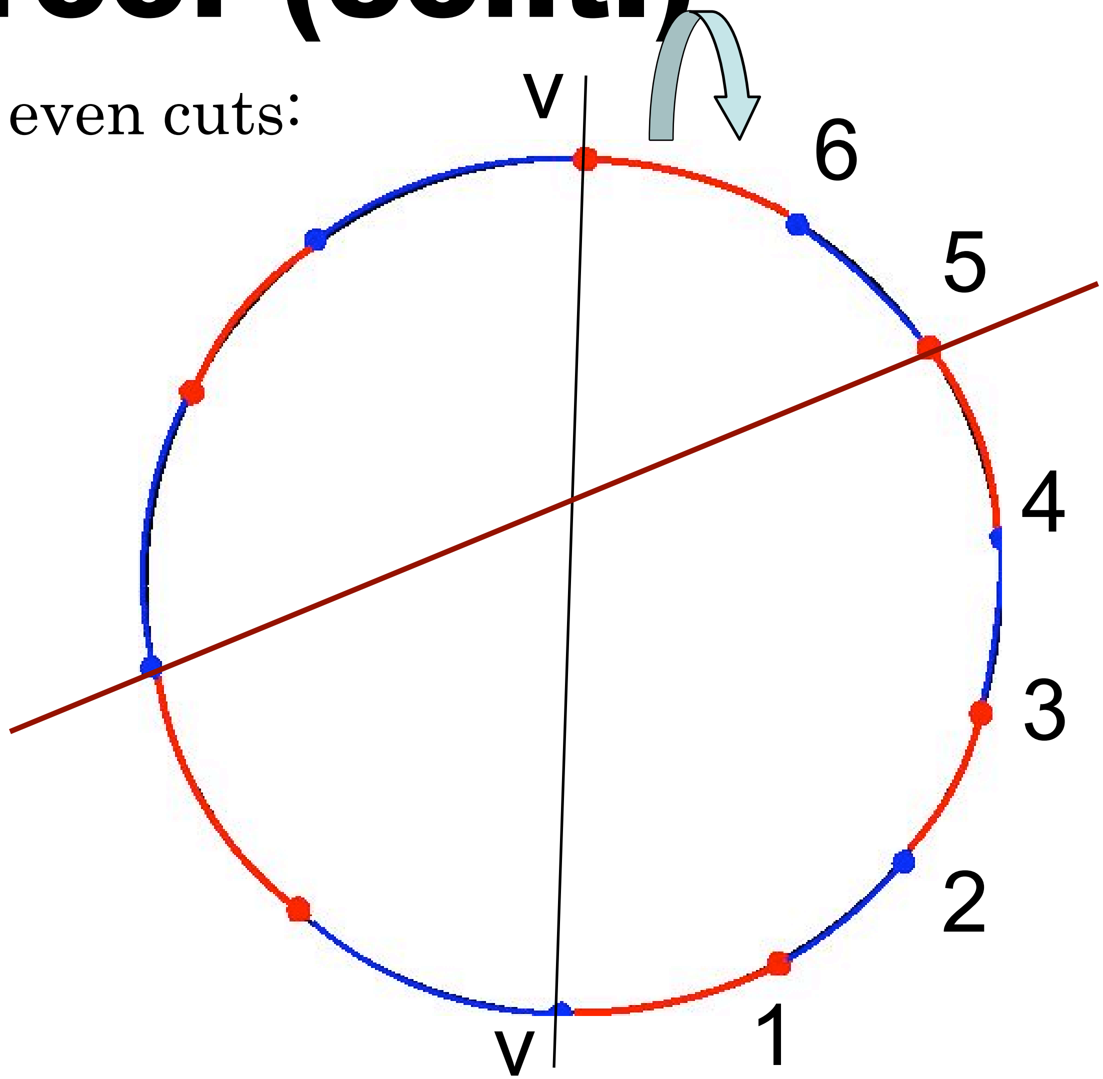

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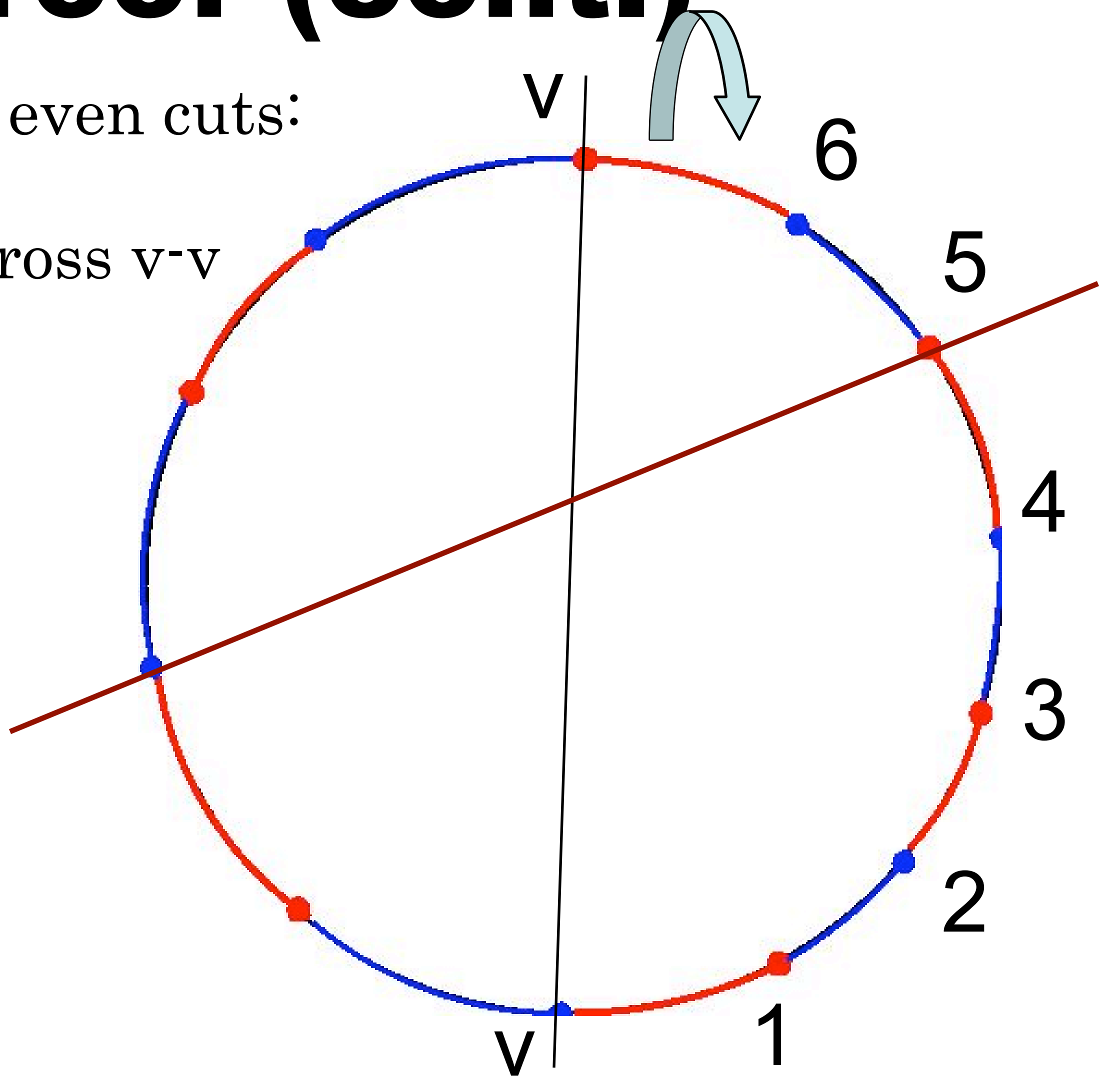

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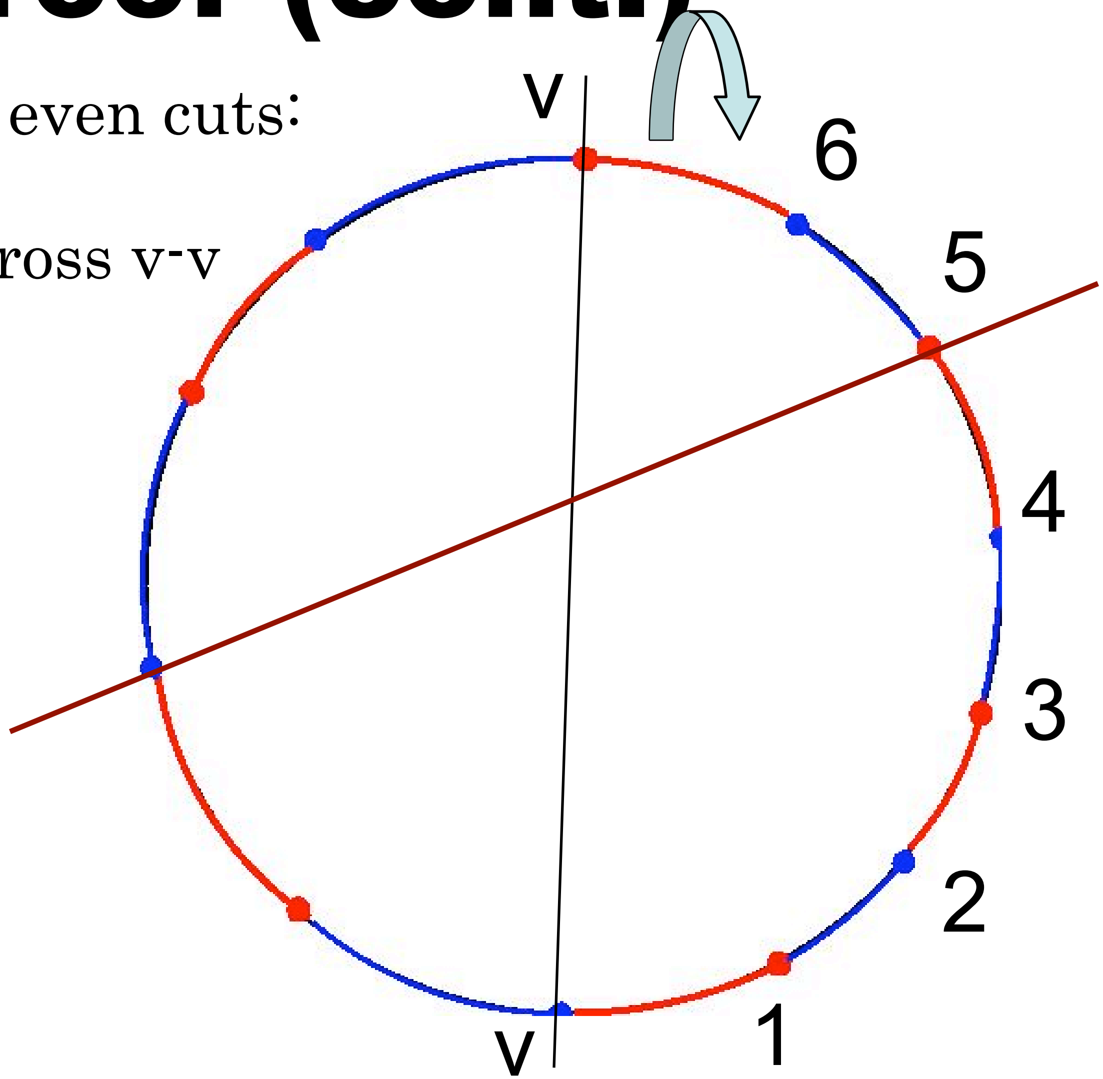

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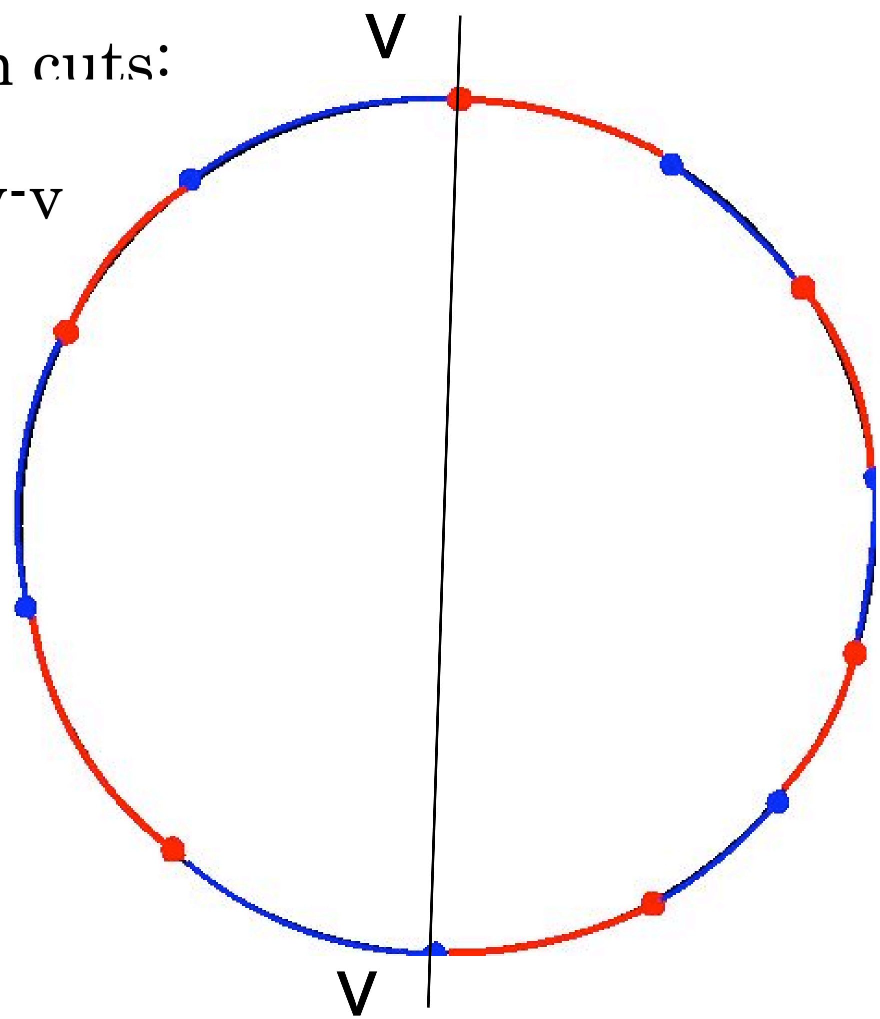

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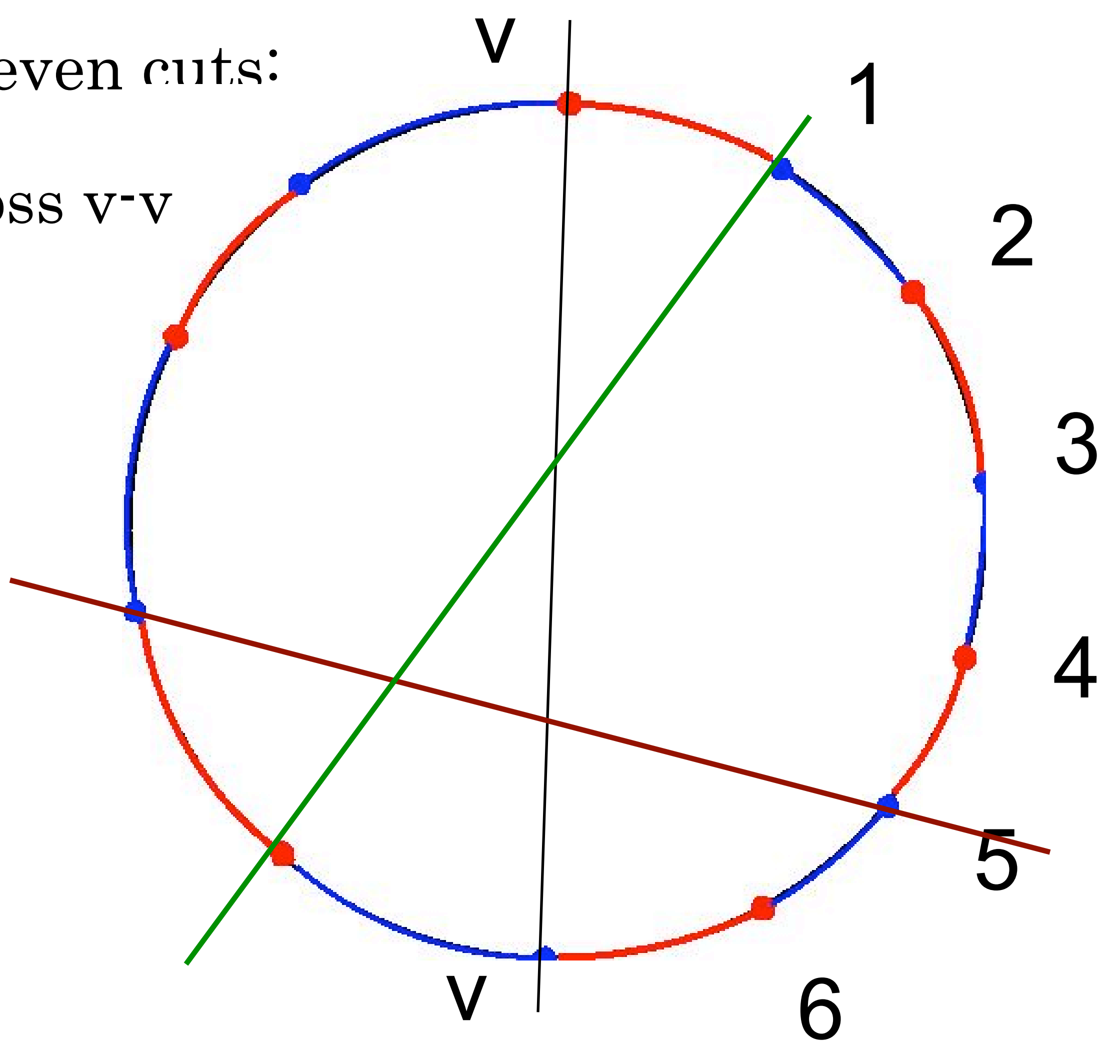

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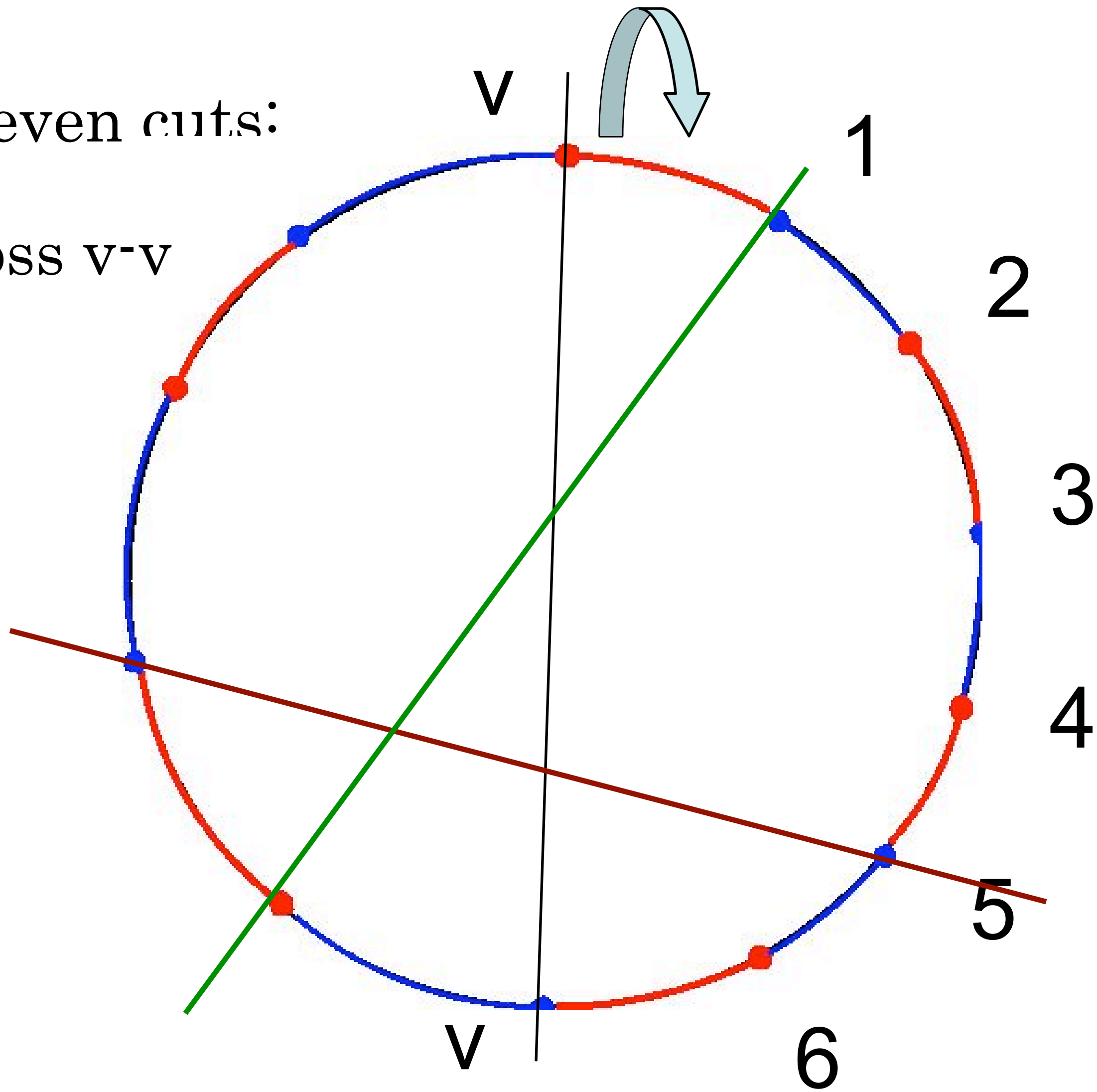

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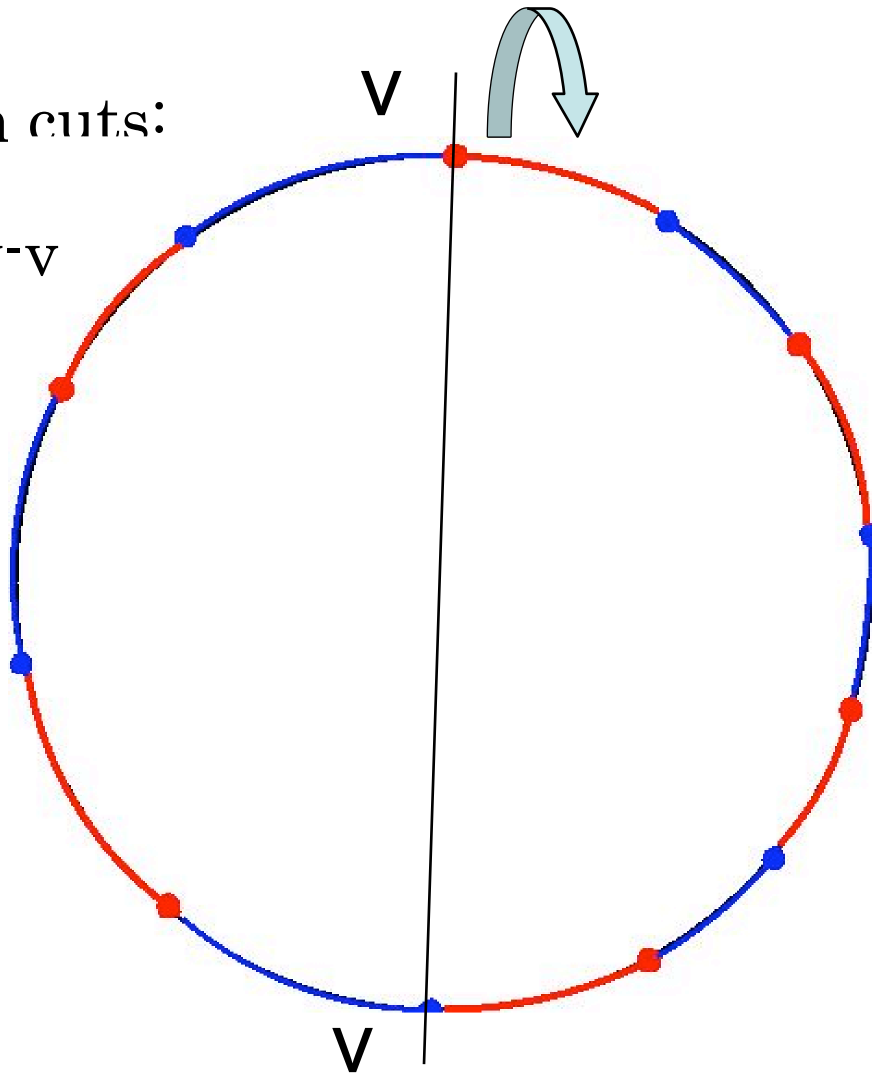

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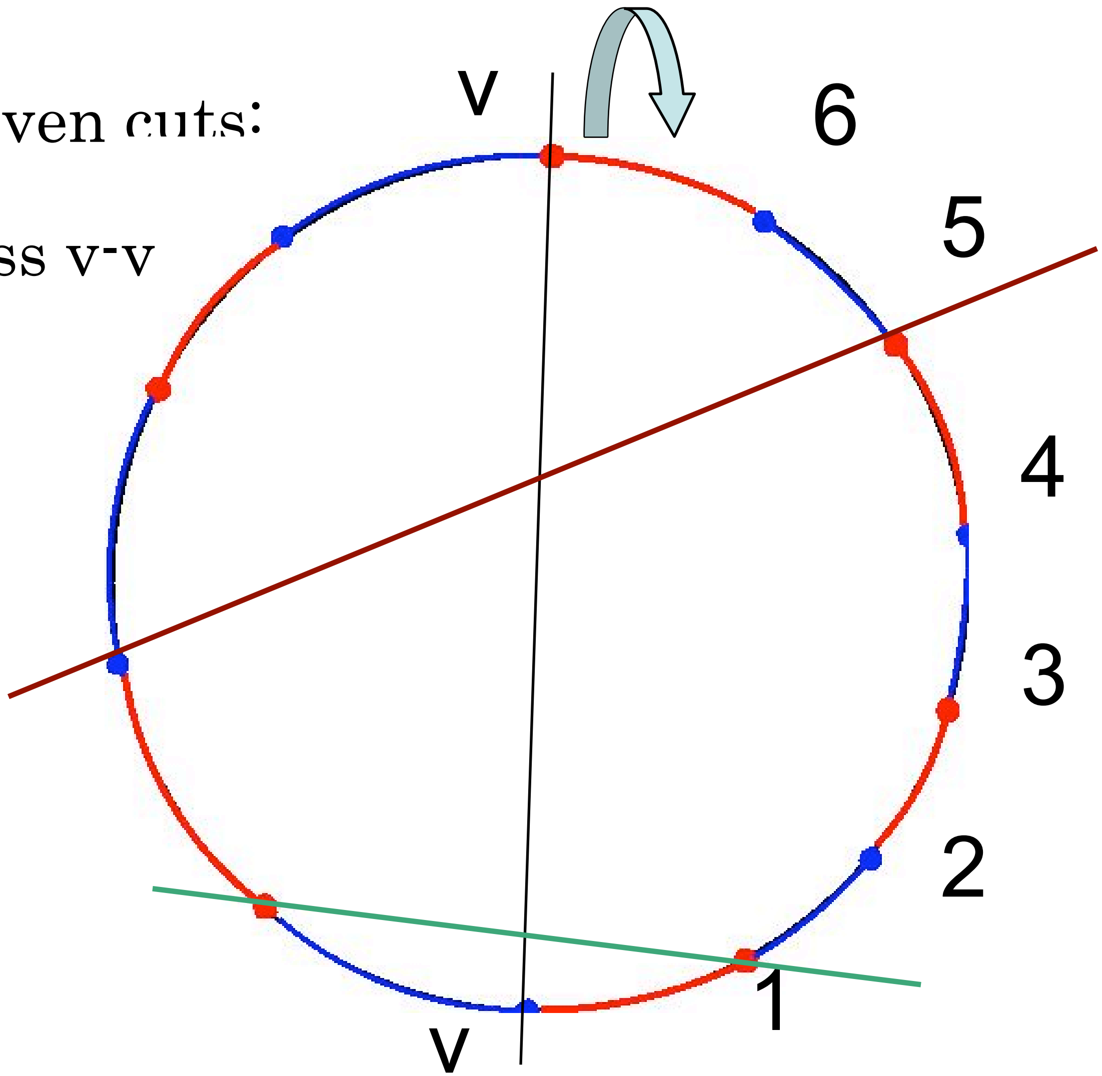

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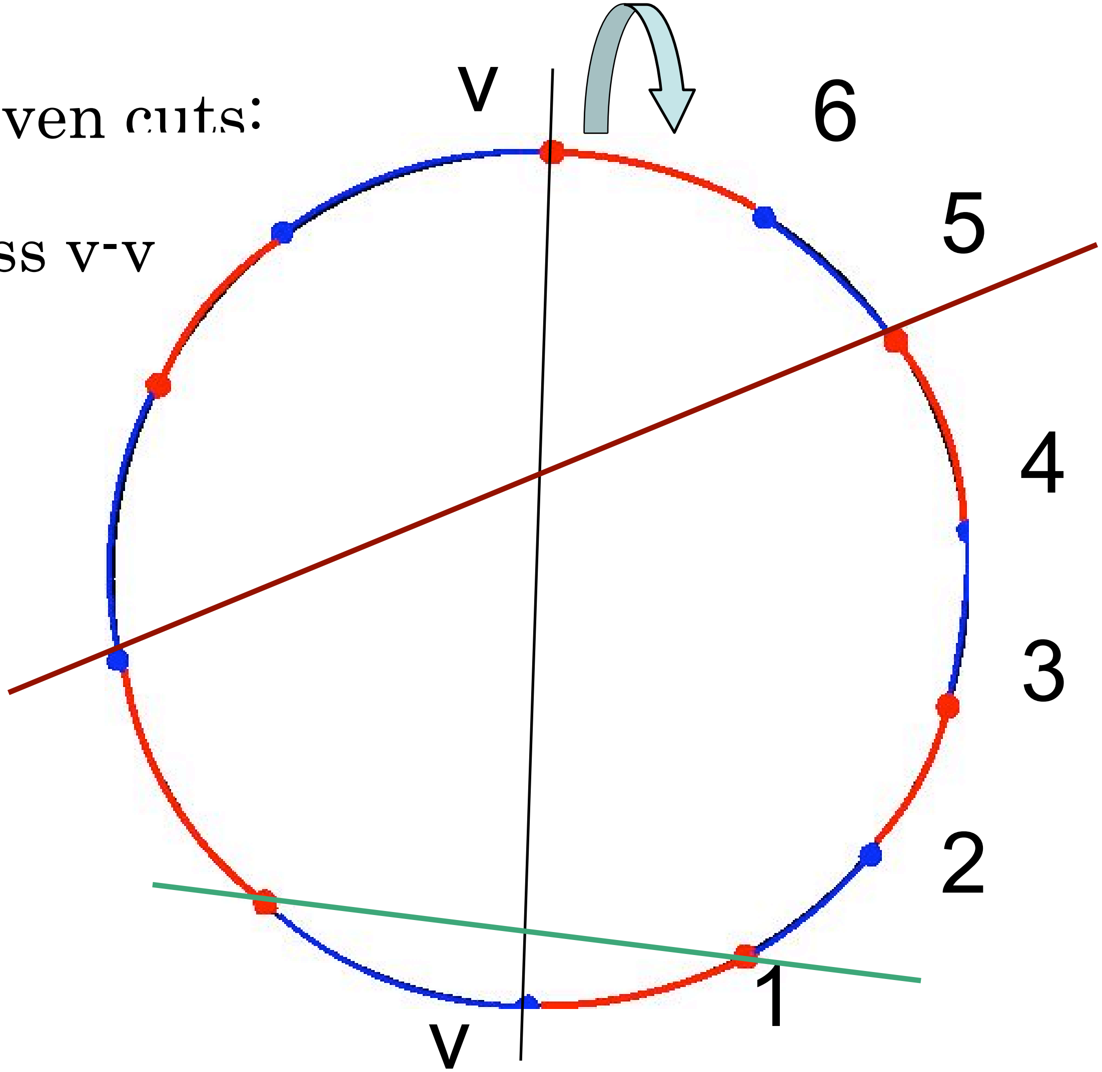

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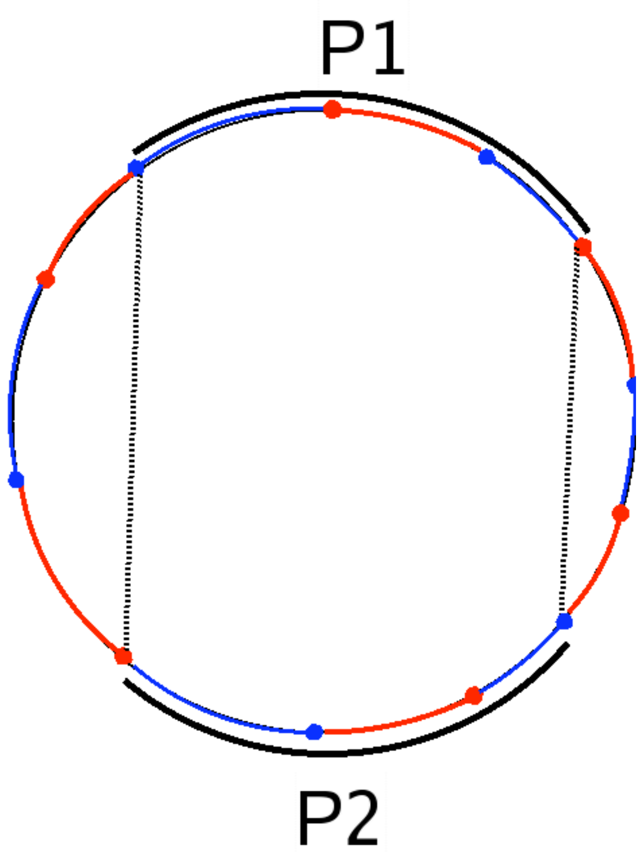

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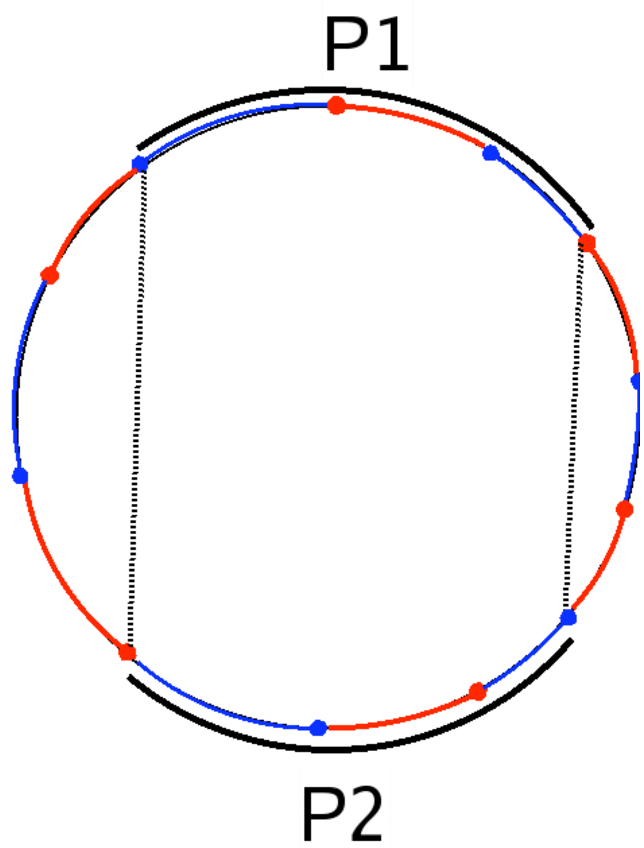
❖Replace P1 by P2 or vice versa



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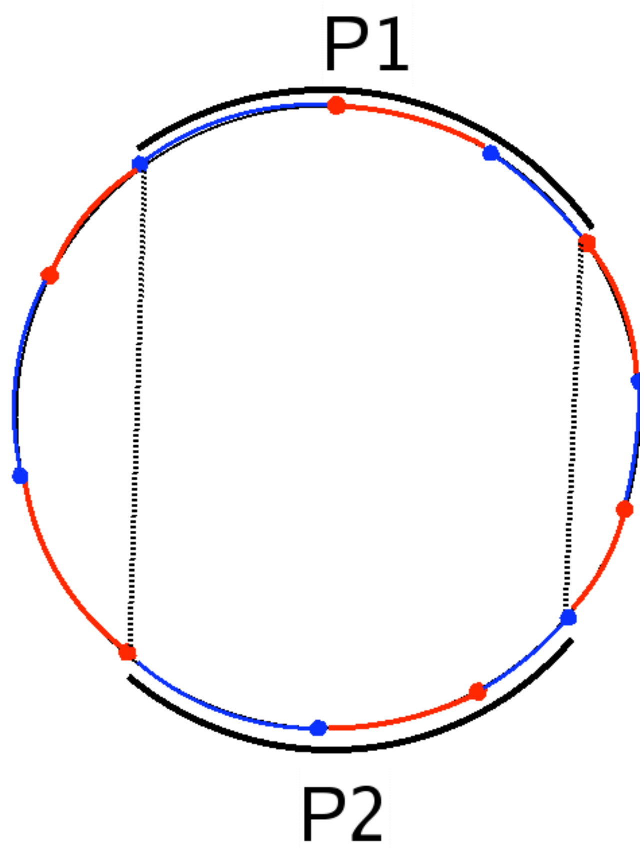
❖Replace P1 by P2 or vice versa

❖Assume they are identical.



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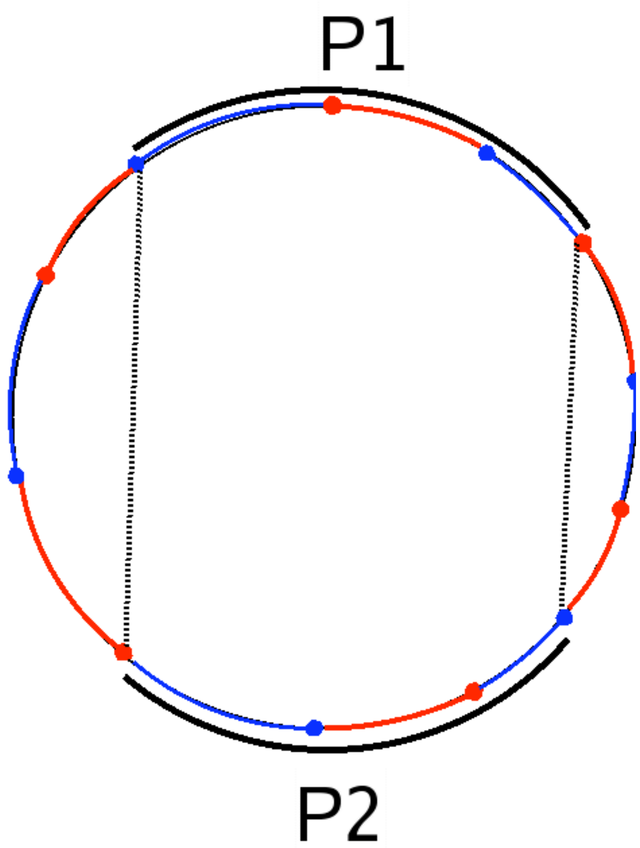

- ❖Replace P1 by P2 or vice versa
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- ❖P1 = P2 , simple and disjoint from the rest of the cycle.

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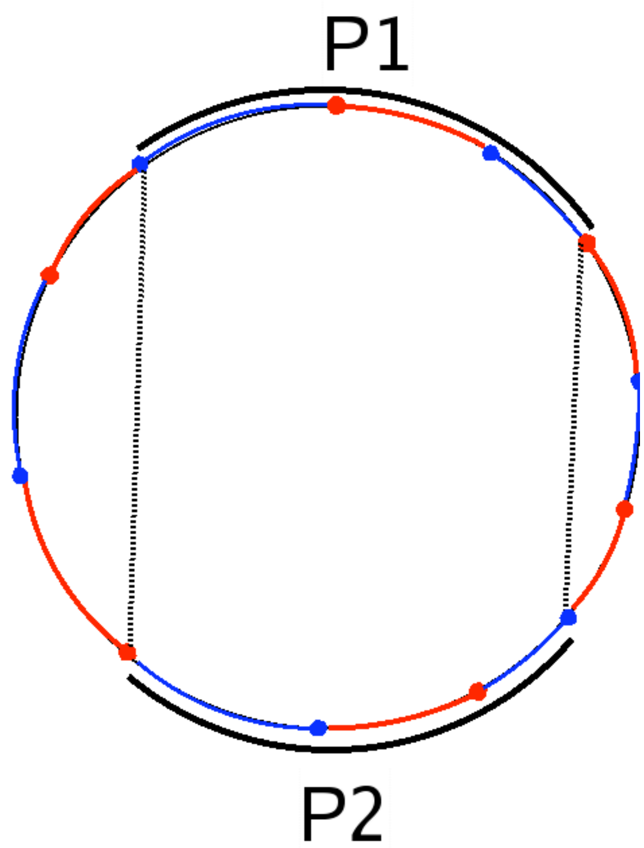

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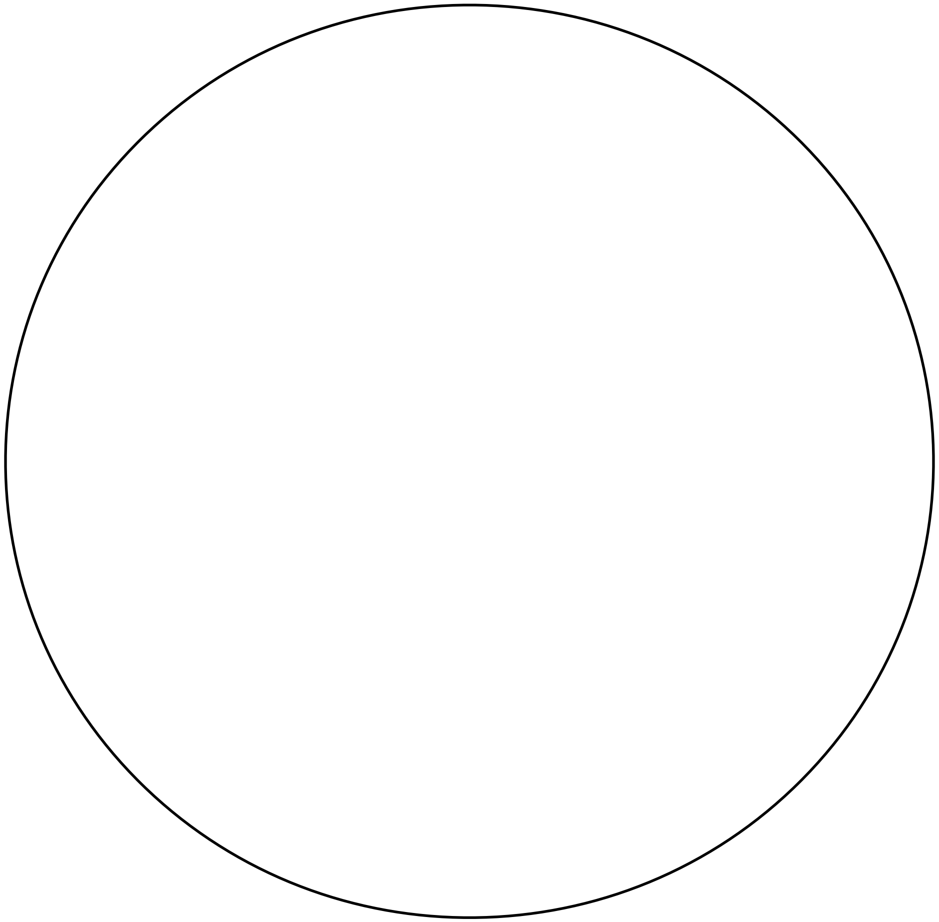
❖Hence the odd cuts are partitioned into blocks

proof (cont.)



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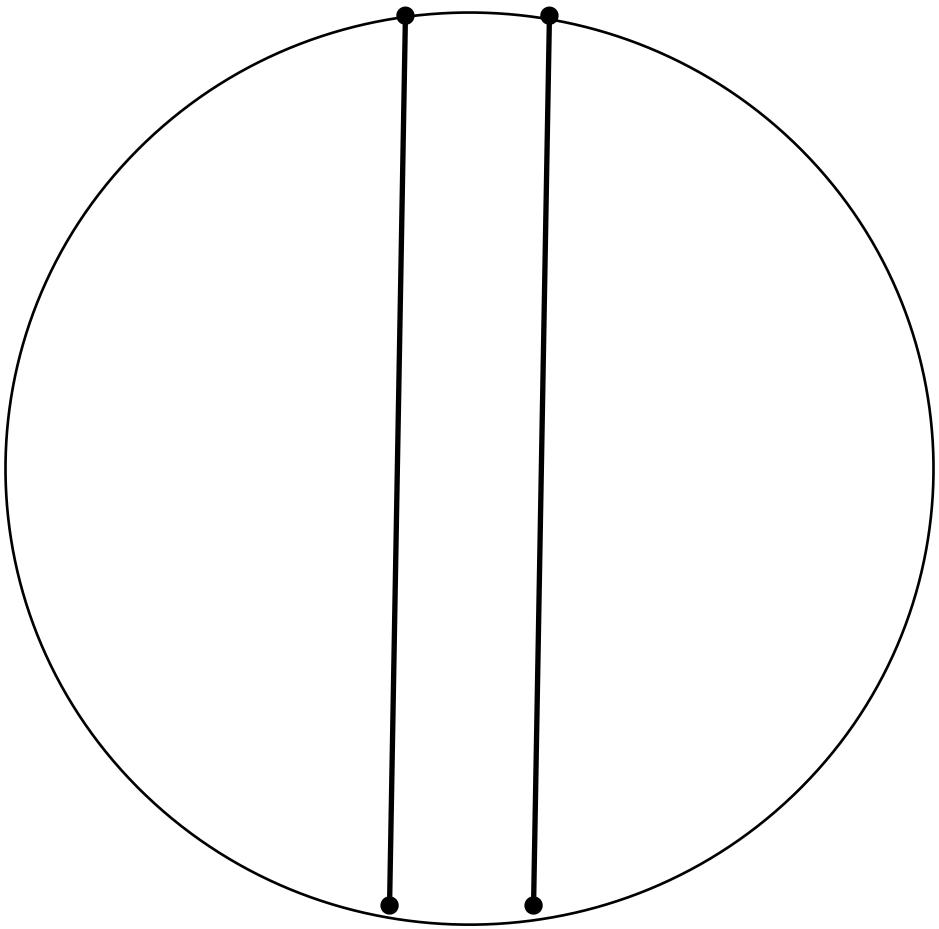

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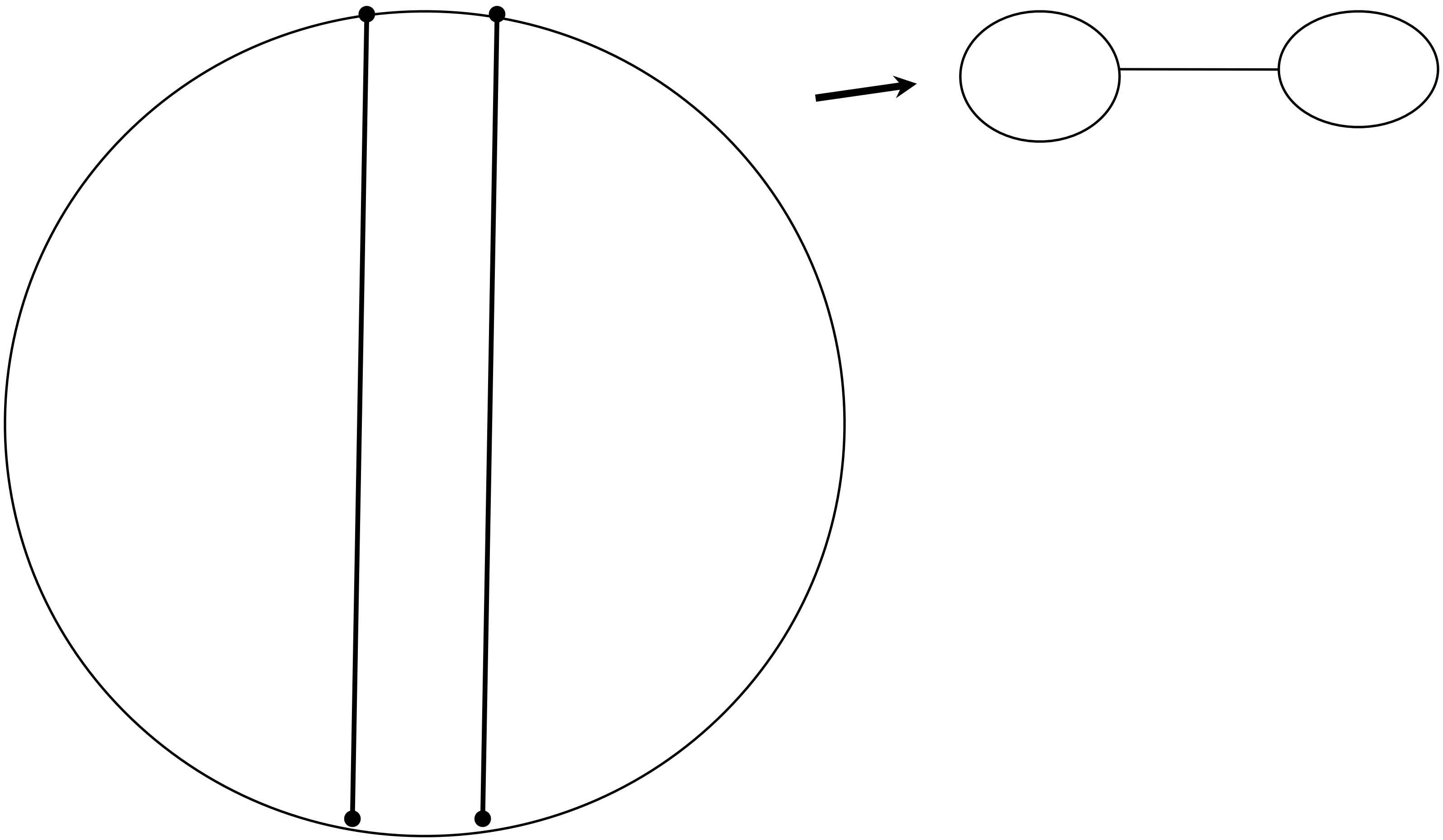

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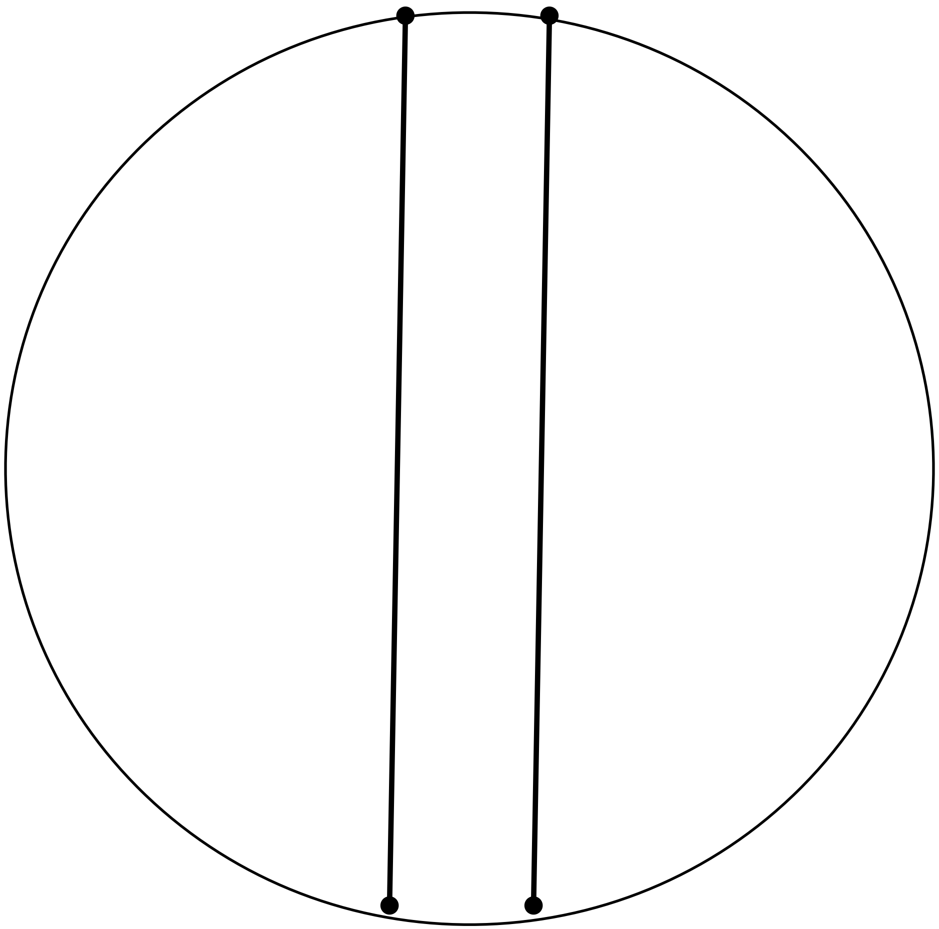

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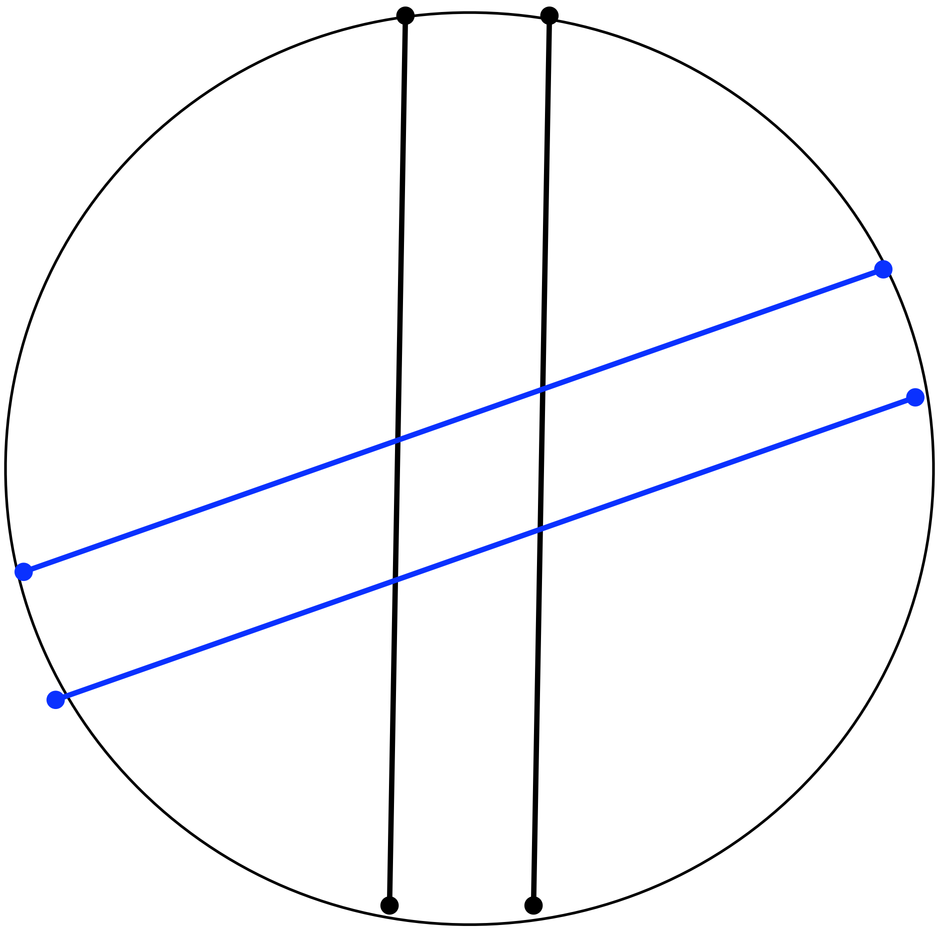

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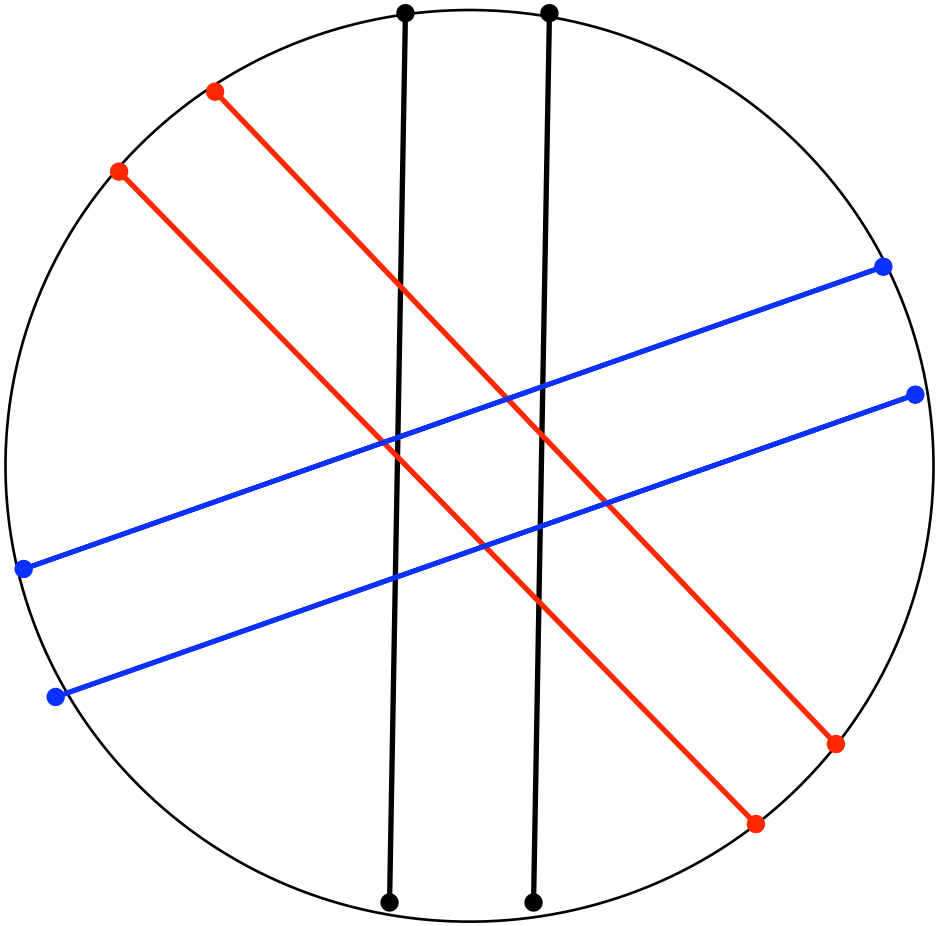

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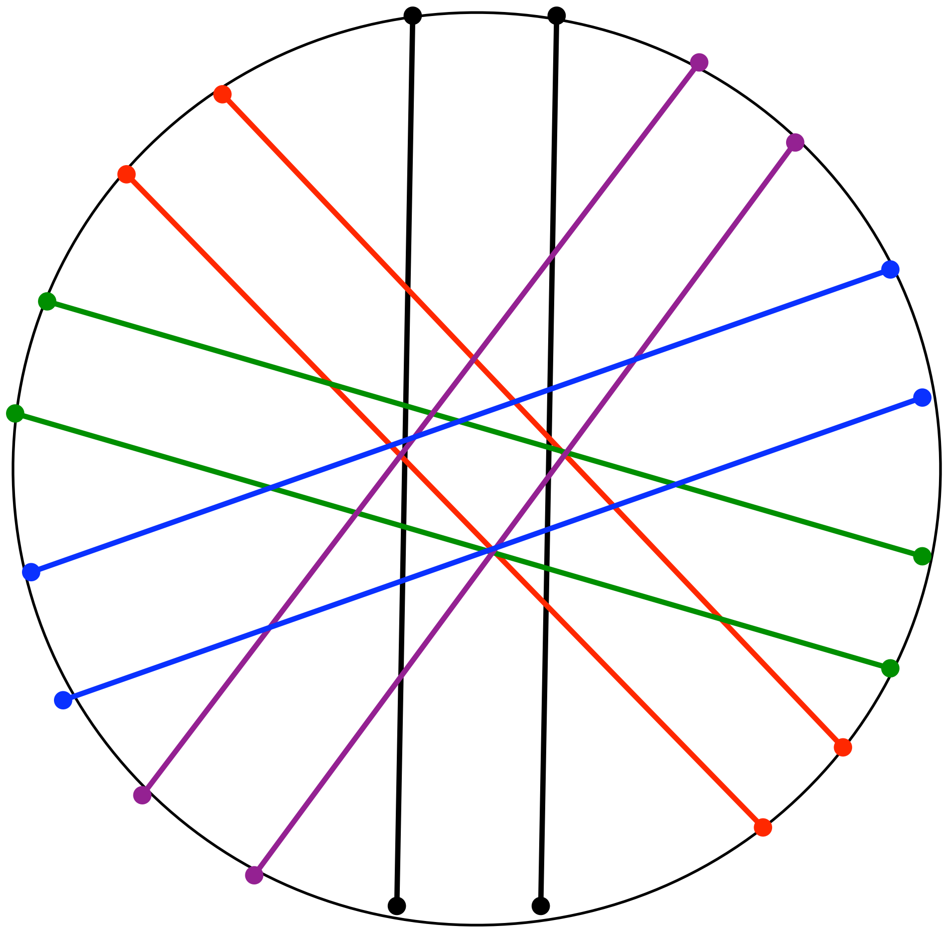

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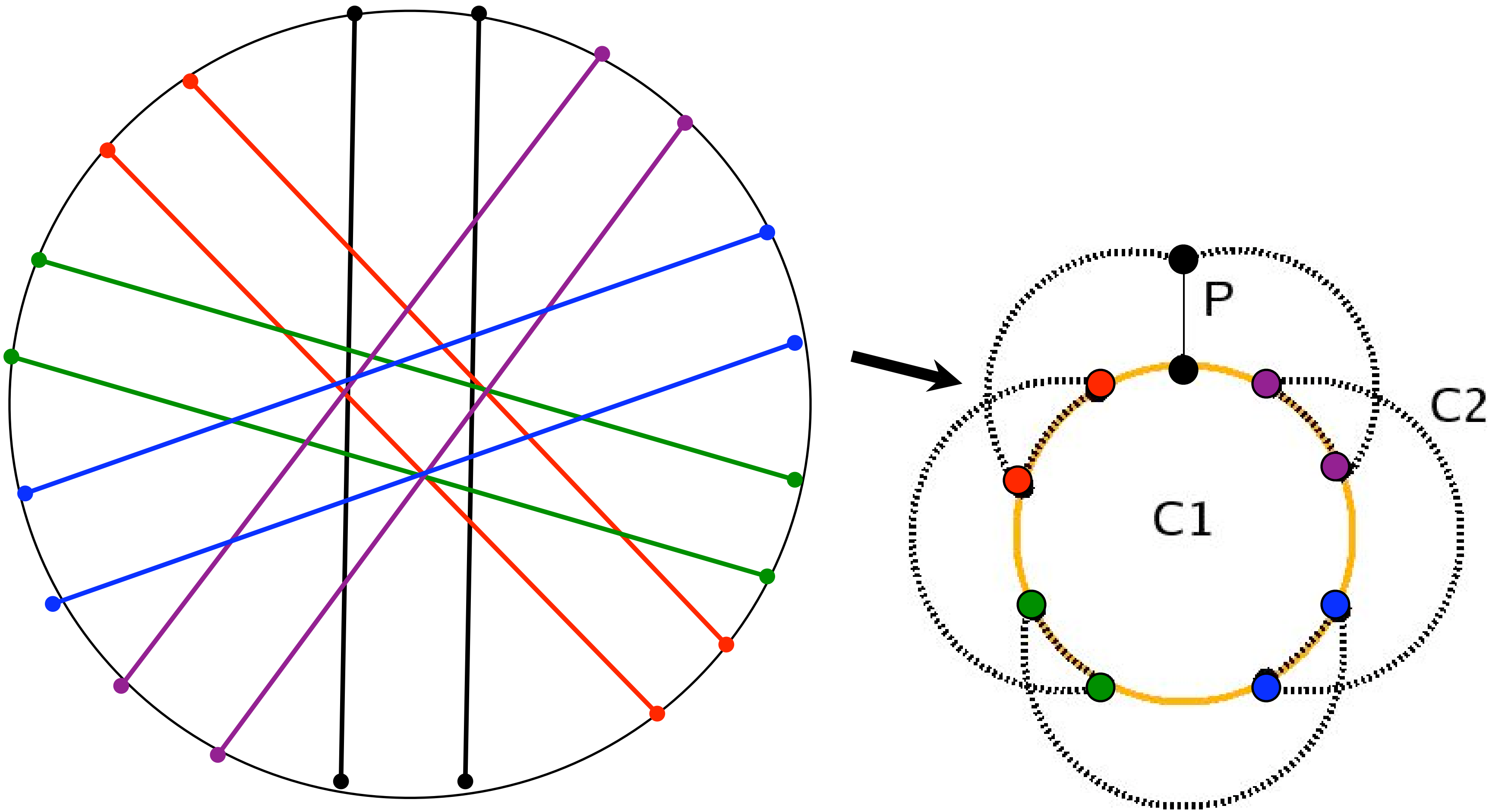
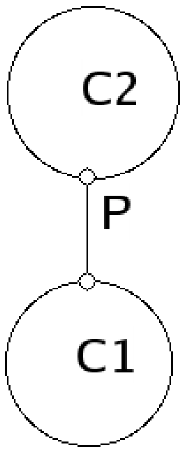
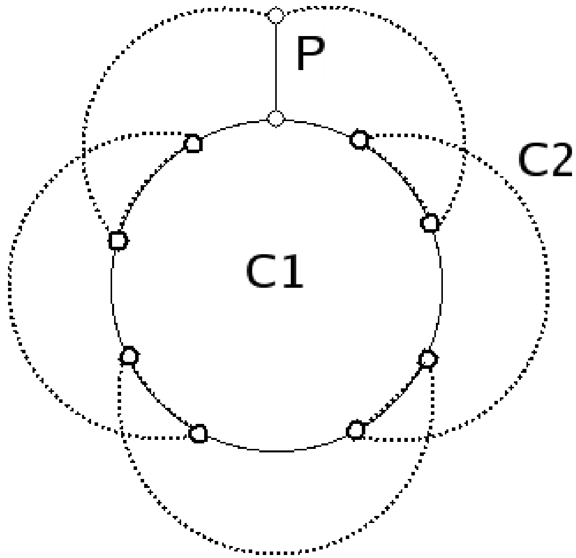
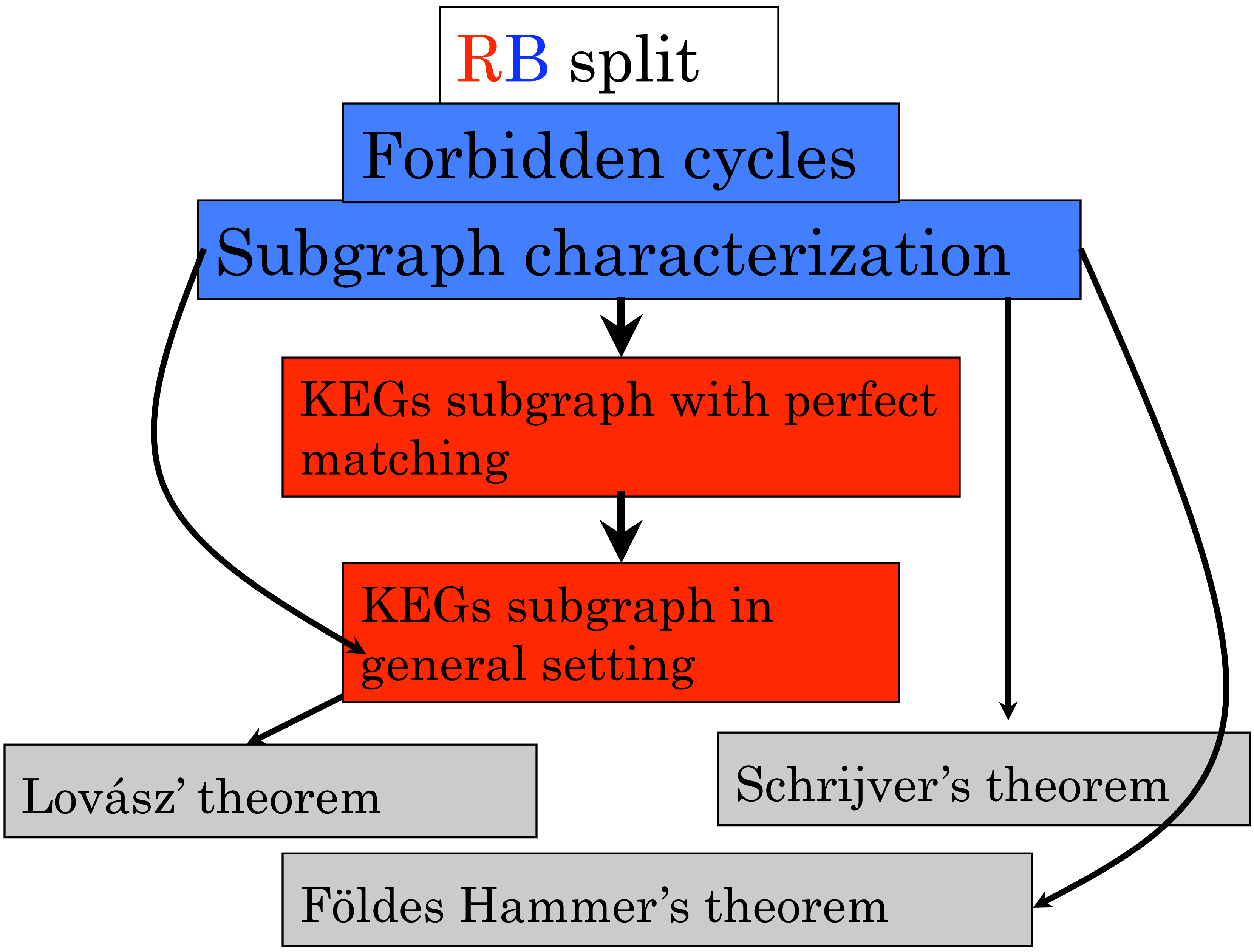


Subgraph characterization of red blue split graphs

✓ A graph is RB split iff it doesn't contain the following "bad" subgraphs.

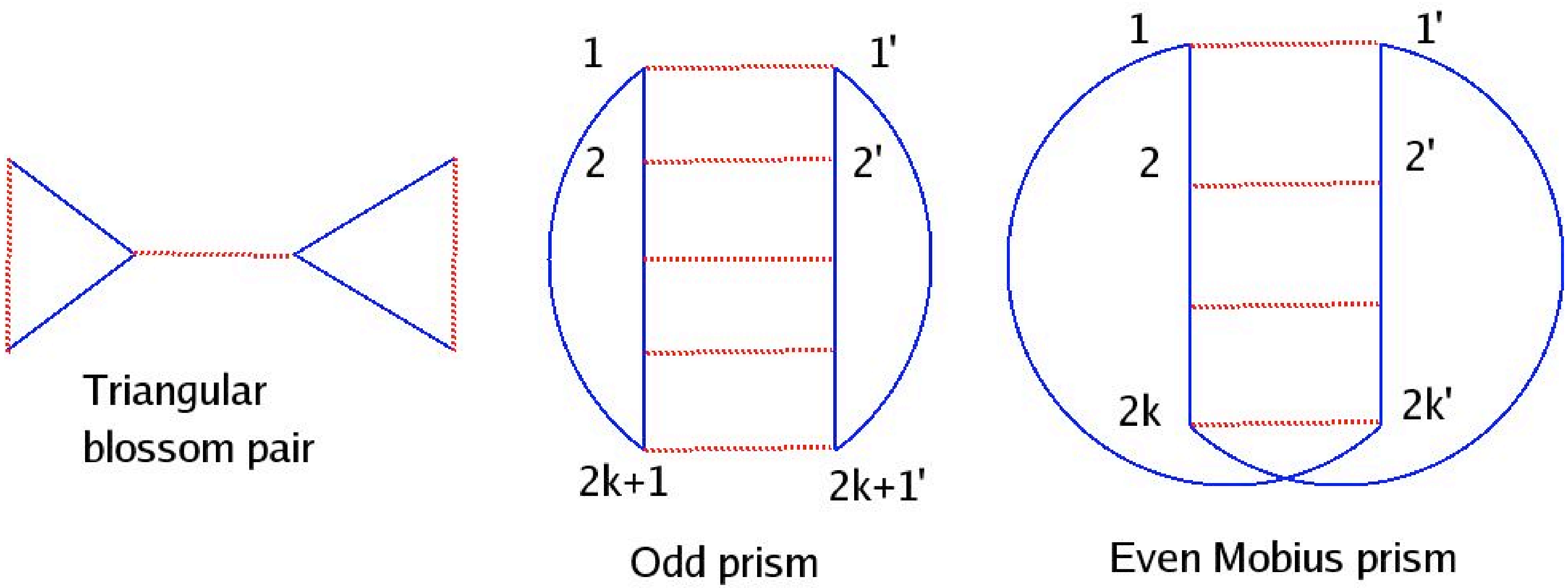





RB split
Forbidden cycles
Subgraph characterization
KEGs subgraph with perfect matching
KEGs subgraph in general setting
Lovász' theorem
Schrijver's theorem
Földes Hammer's theorem

Subgraph characterization of KEGs with perfect matching

❖Given a graph with perfect matching, it is KEG iff does not contain:



Subgraph characterization of KEGs with perfect matching

- We showed in a blue graph, if the perfect matching is red, then KEG=RB split.

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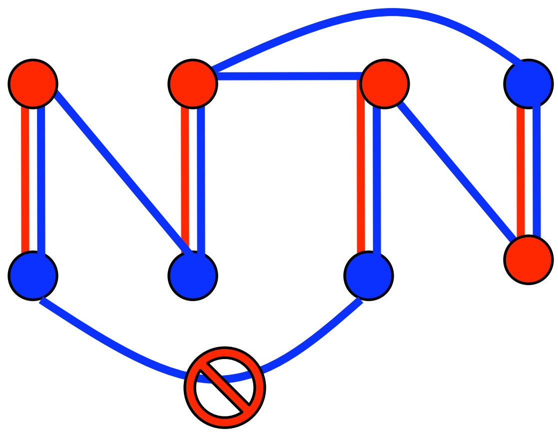

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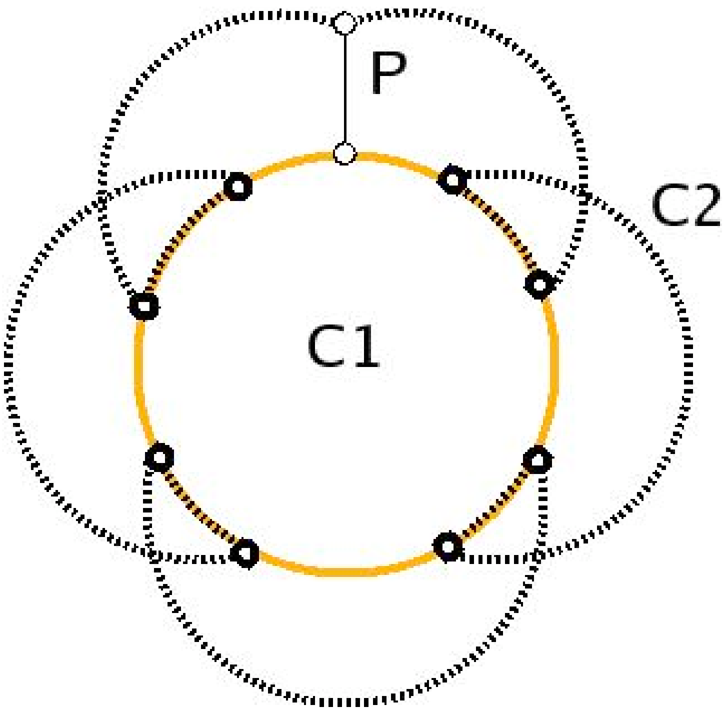

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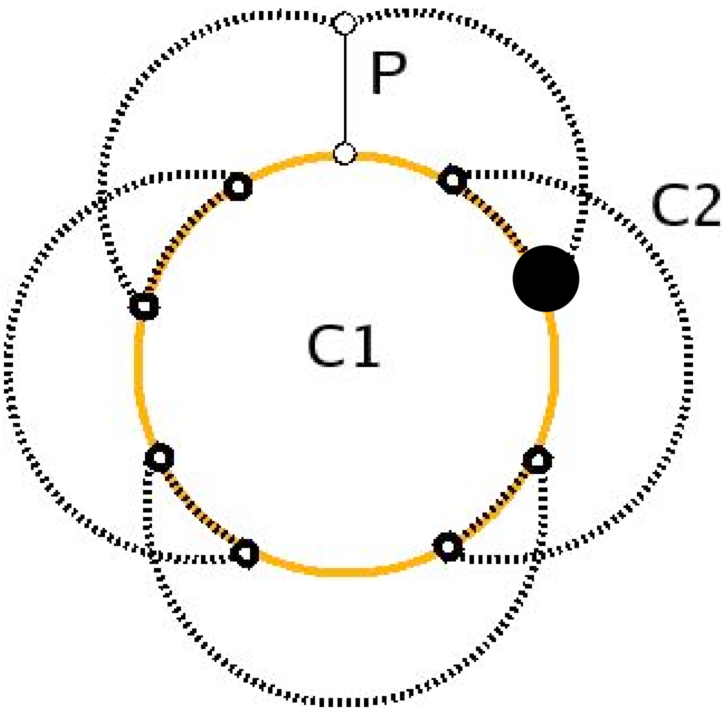

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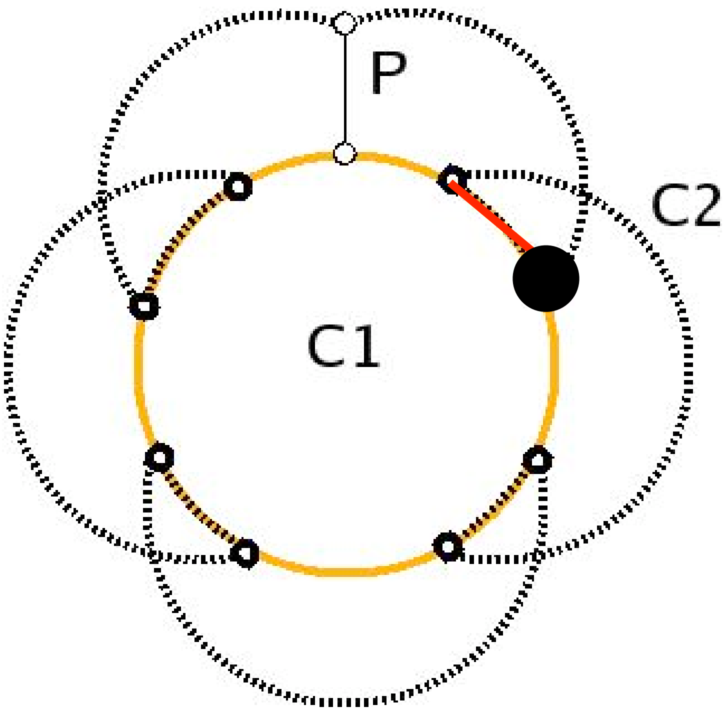

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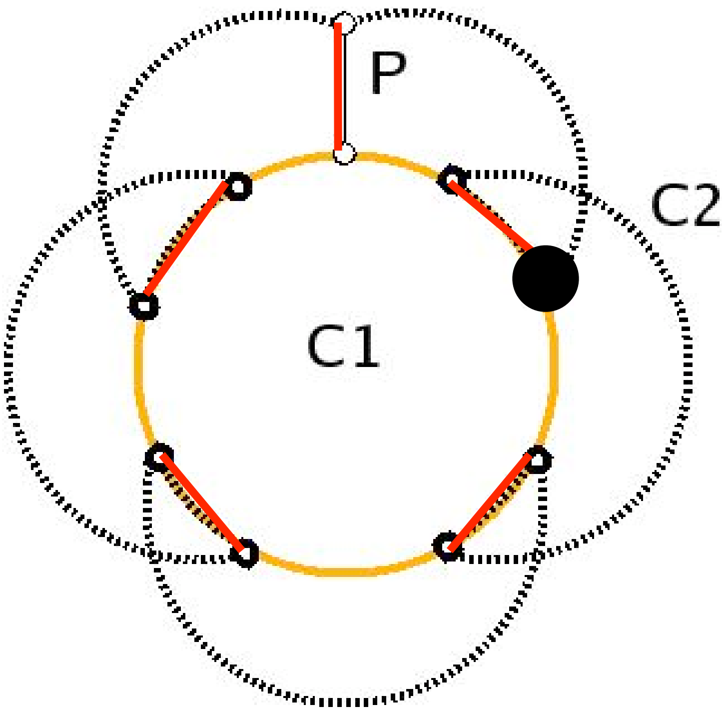

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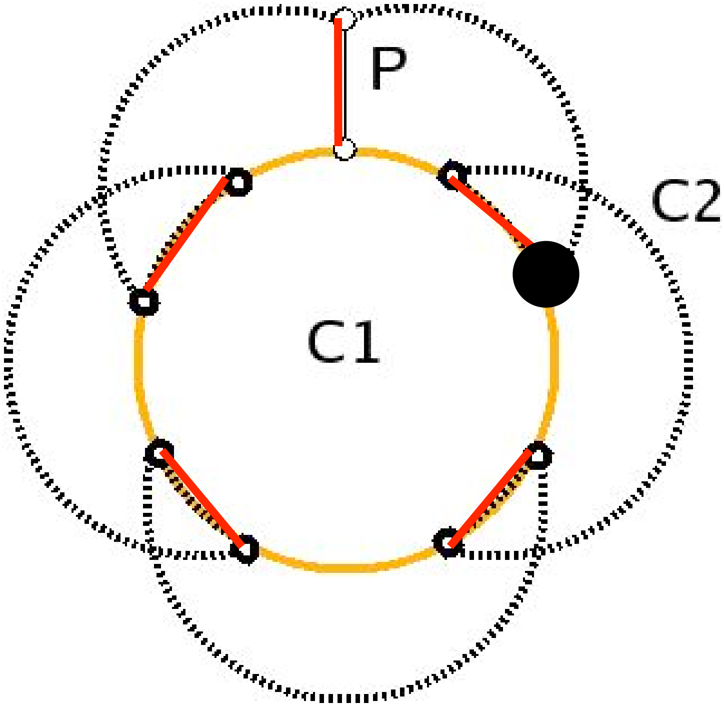

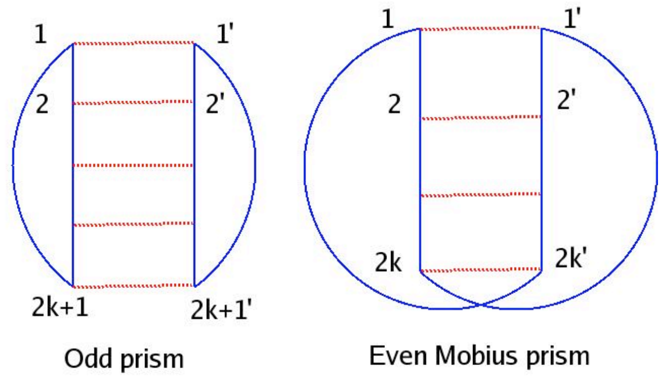

KEGs in general setting

- Given a graph and a maximum matching, it is KEG iff doesn't contain:

KEGs in general setting

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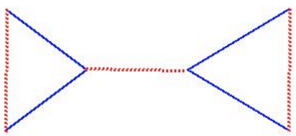
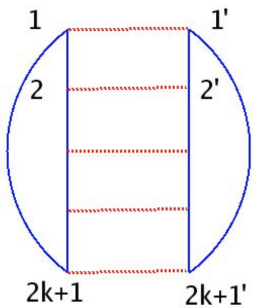
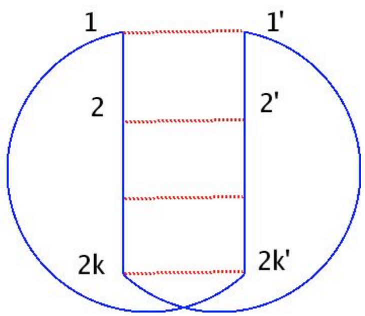


Triangular blossom pair



Odd prism



Even Mobius prism

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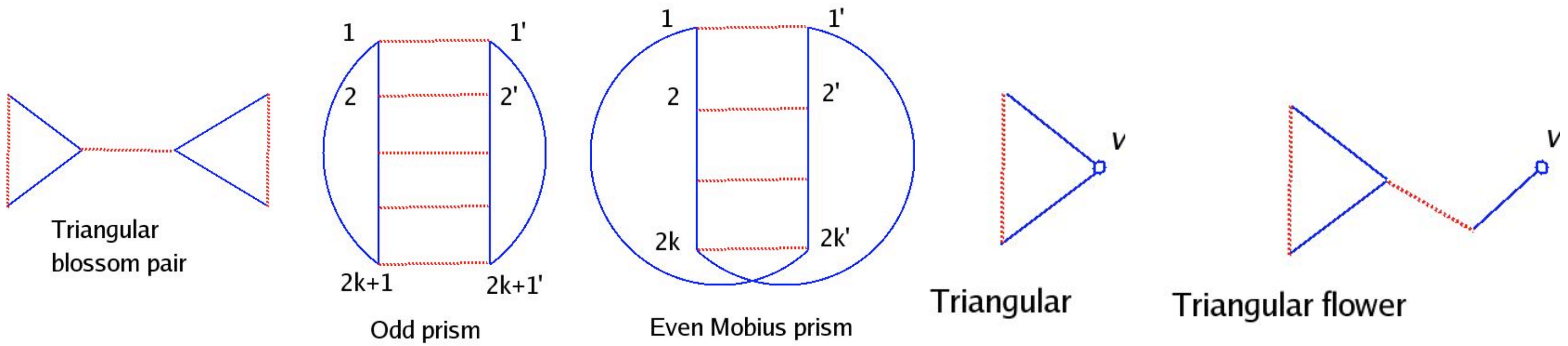

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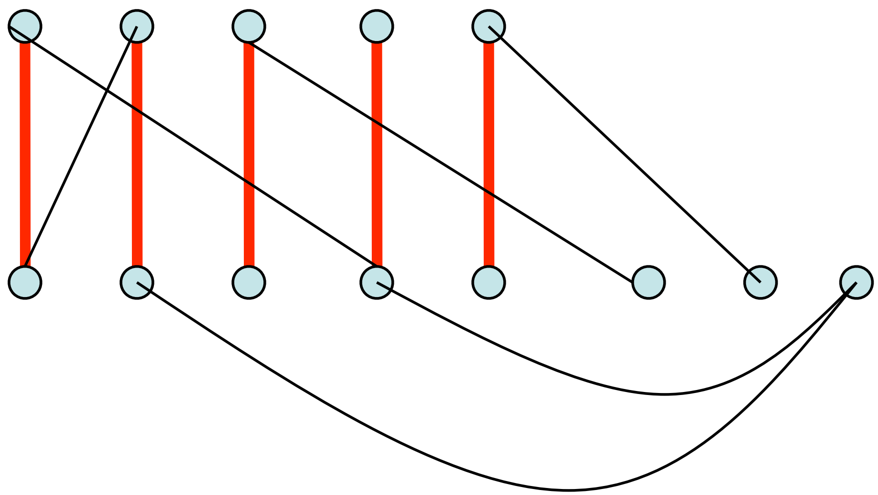

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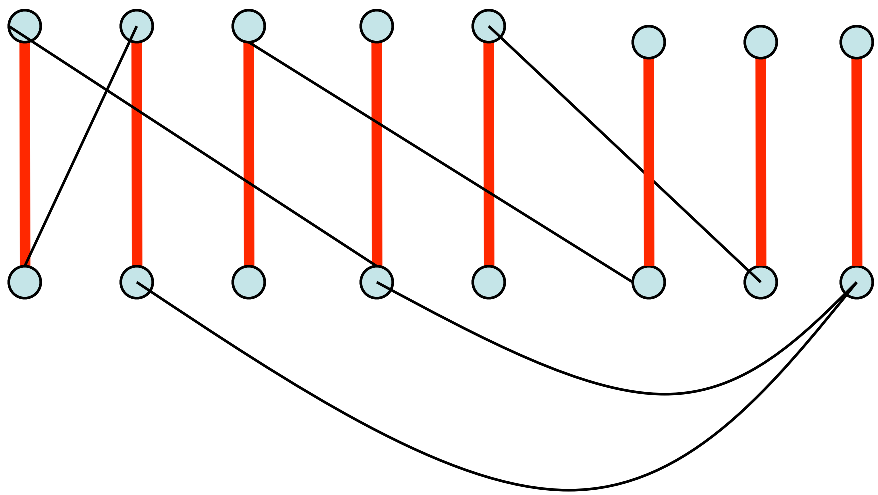

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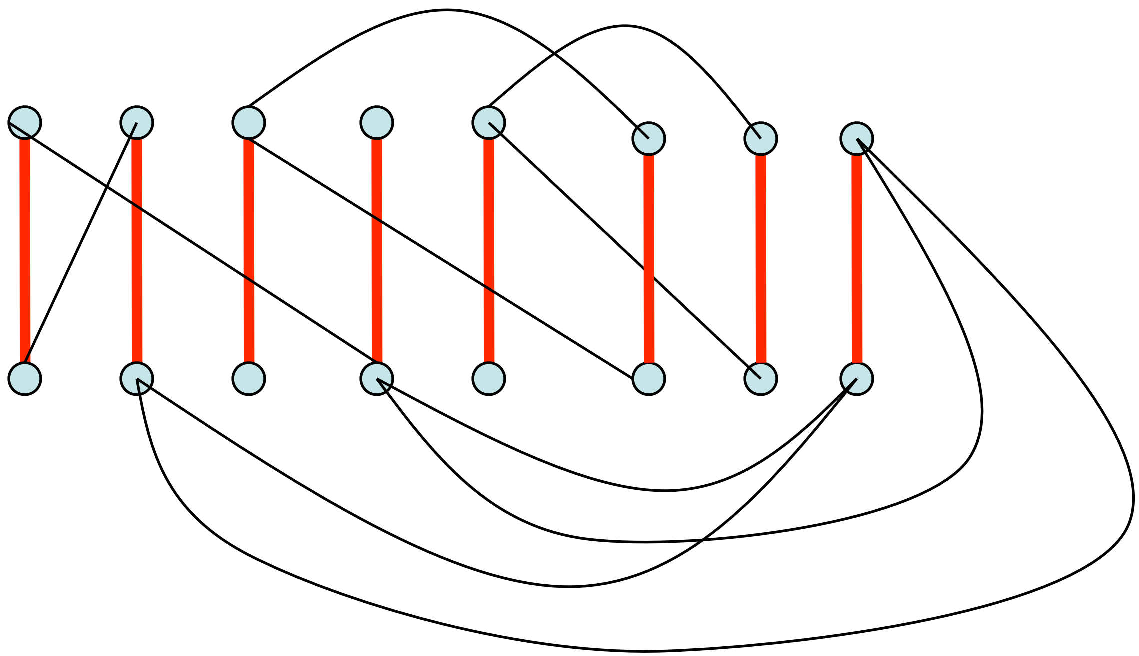

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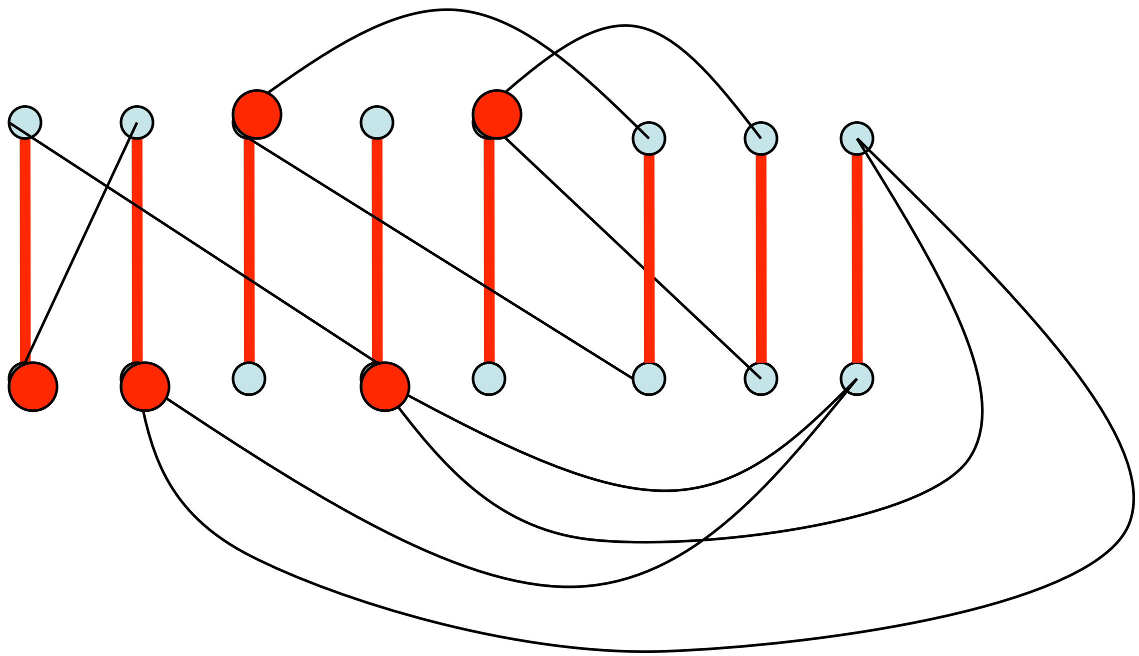

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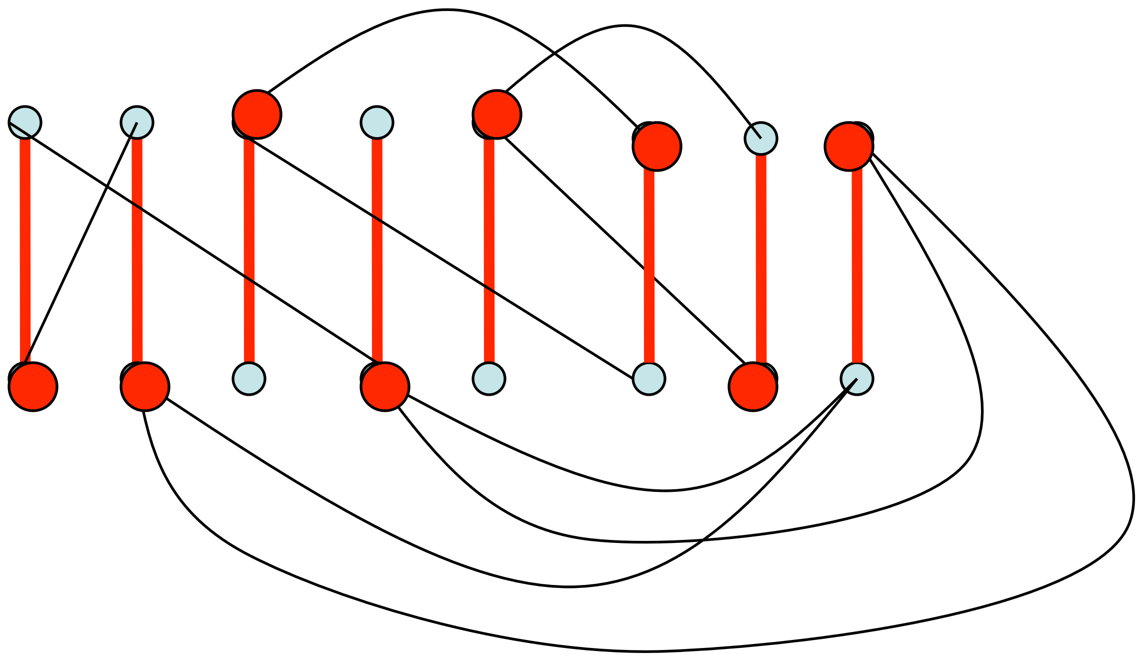

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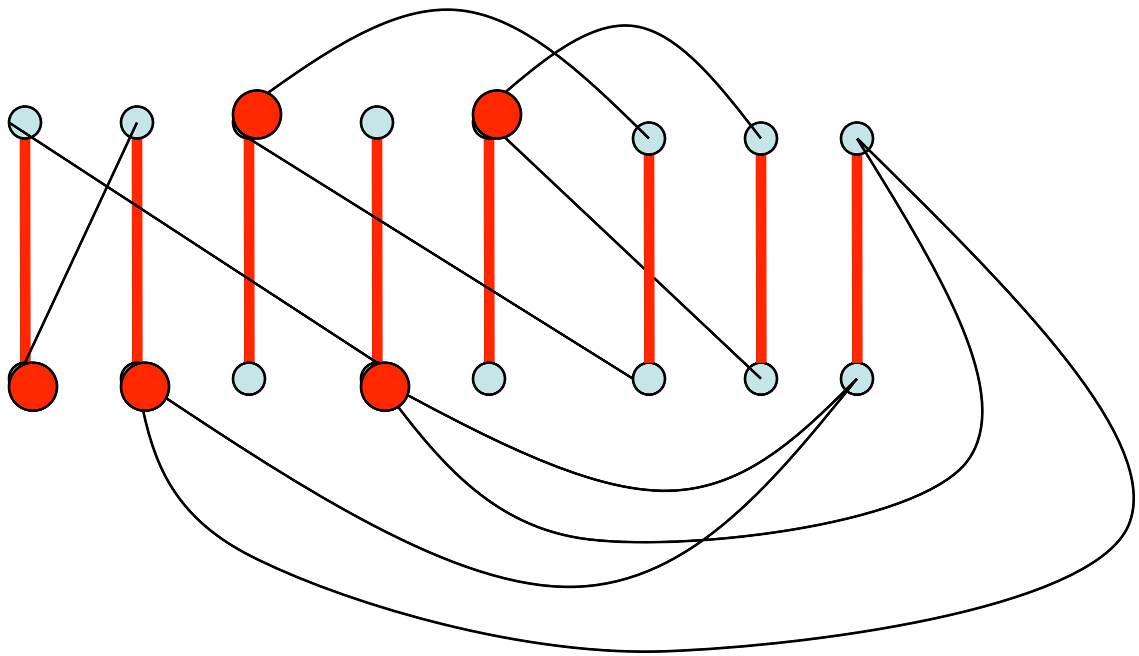

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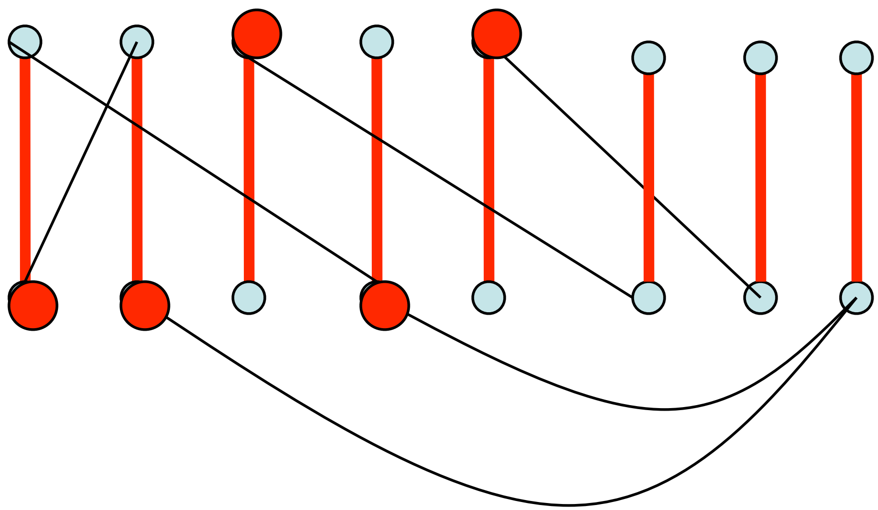

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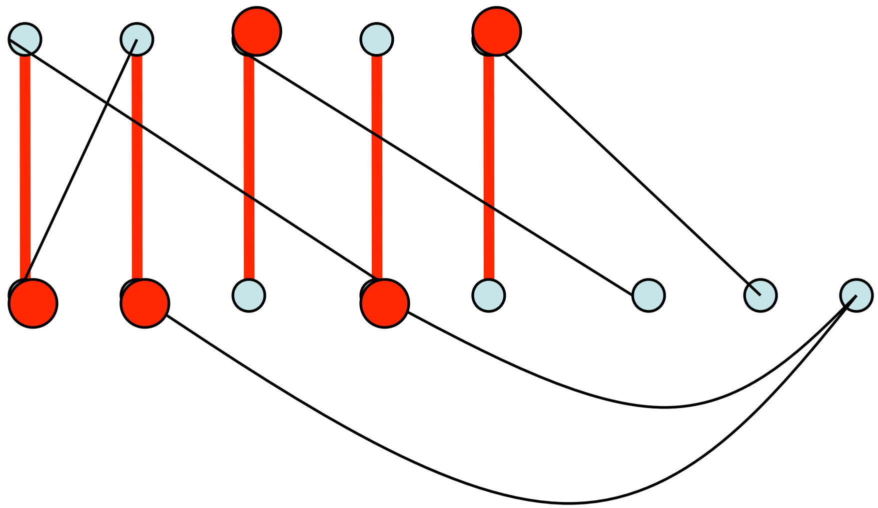

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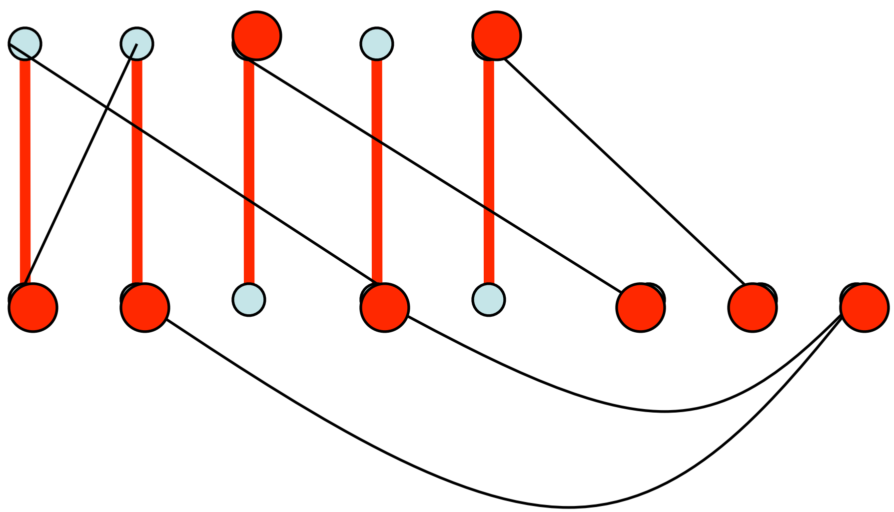

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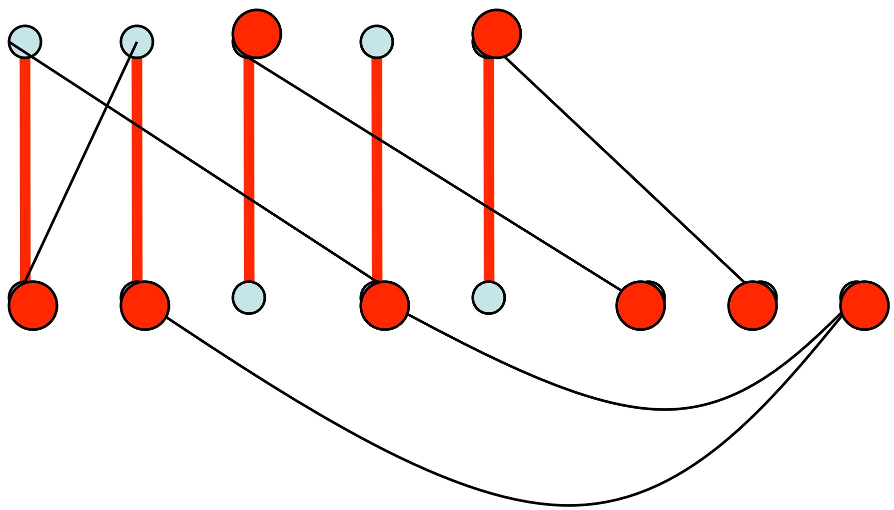

❖ The new graph is KEG iff the original is also.

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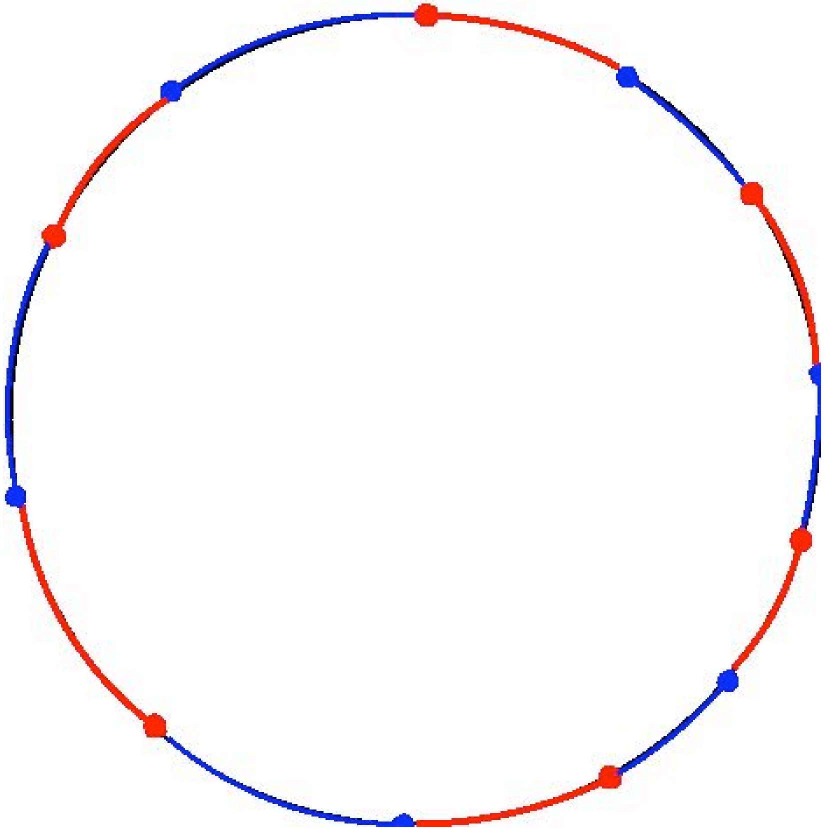

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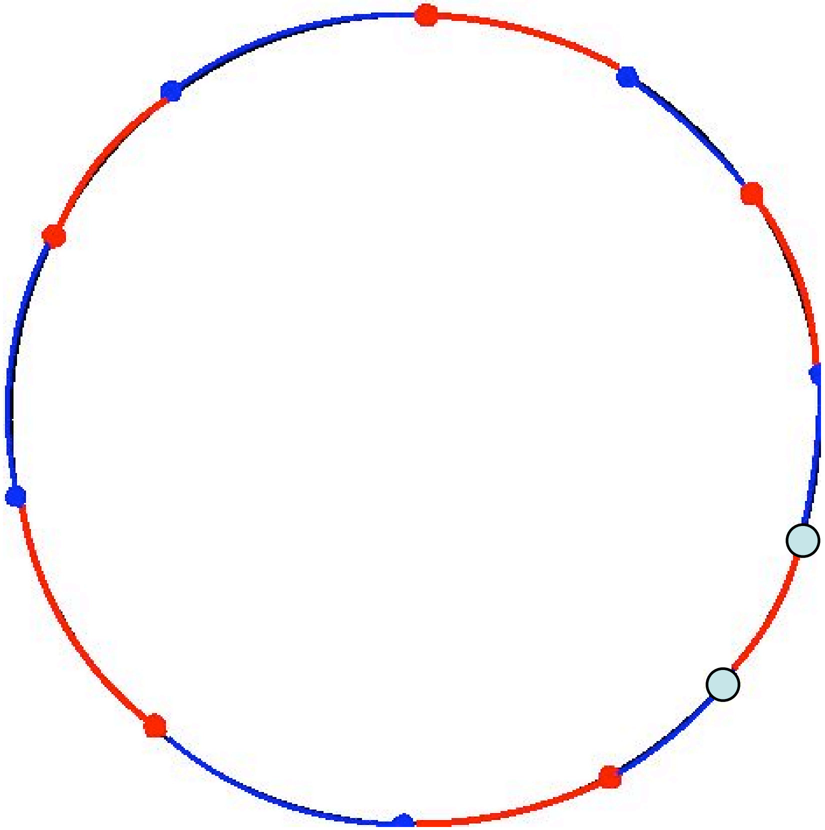

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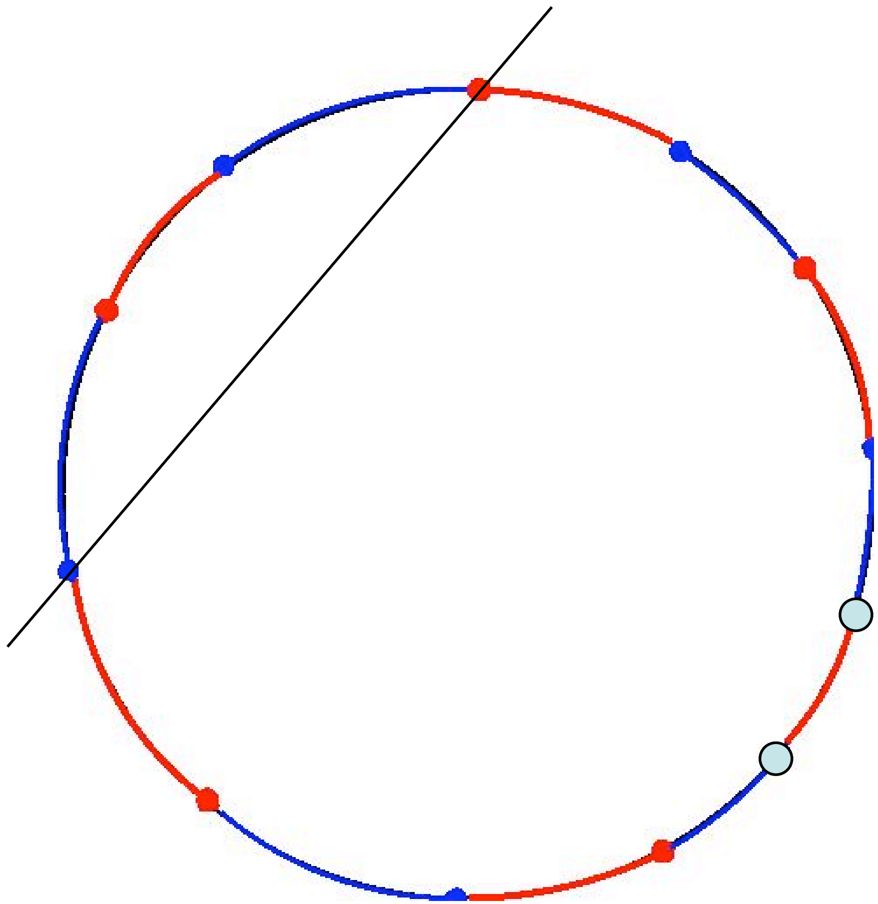

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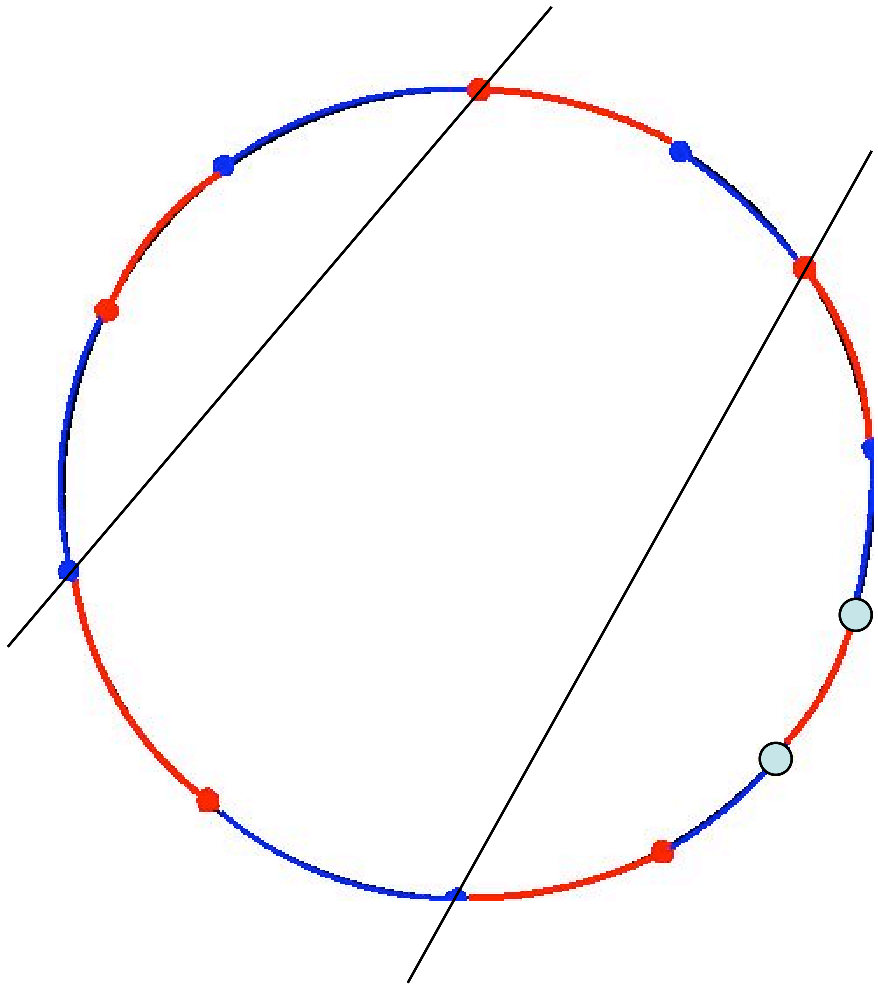

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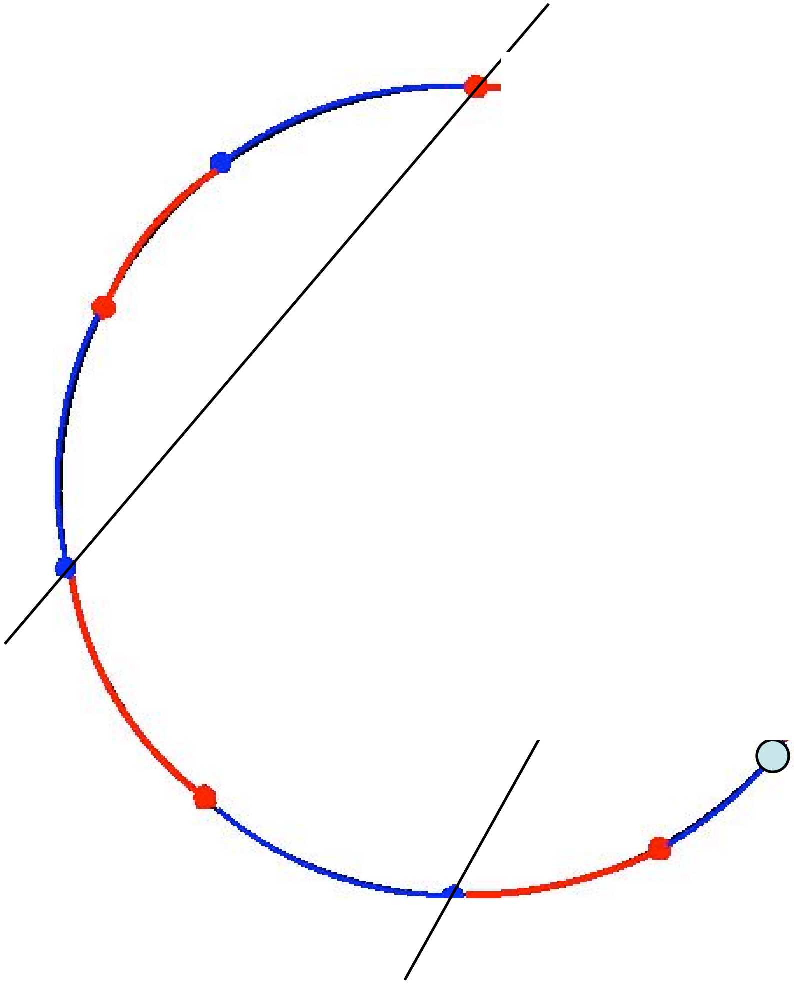

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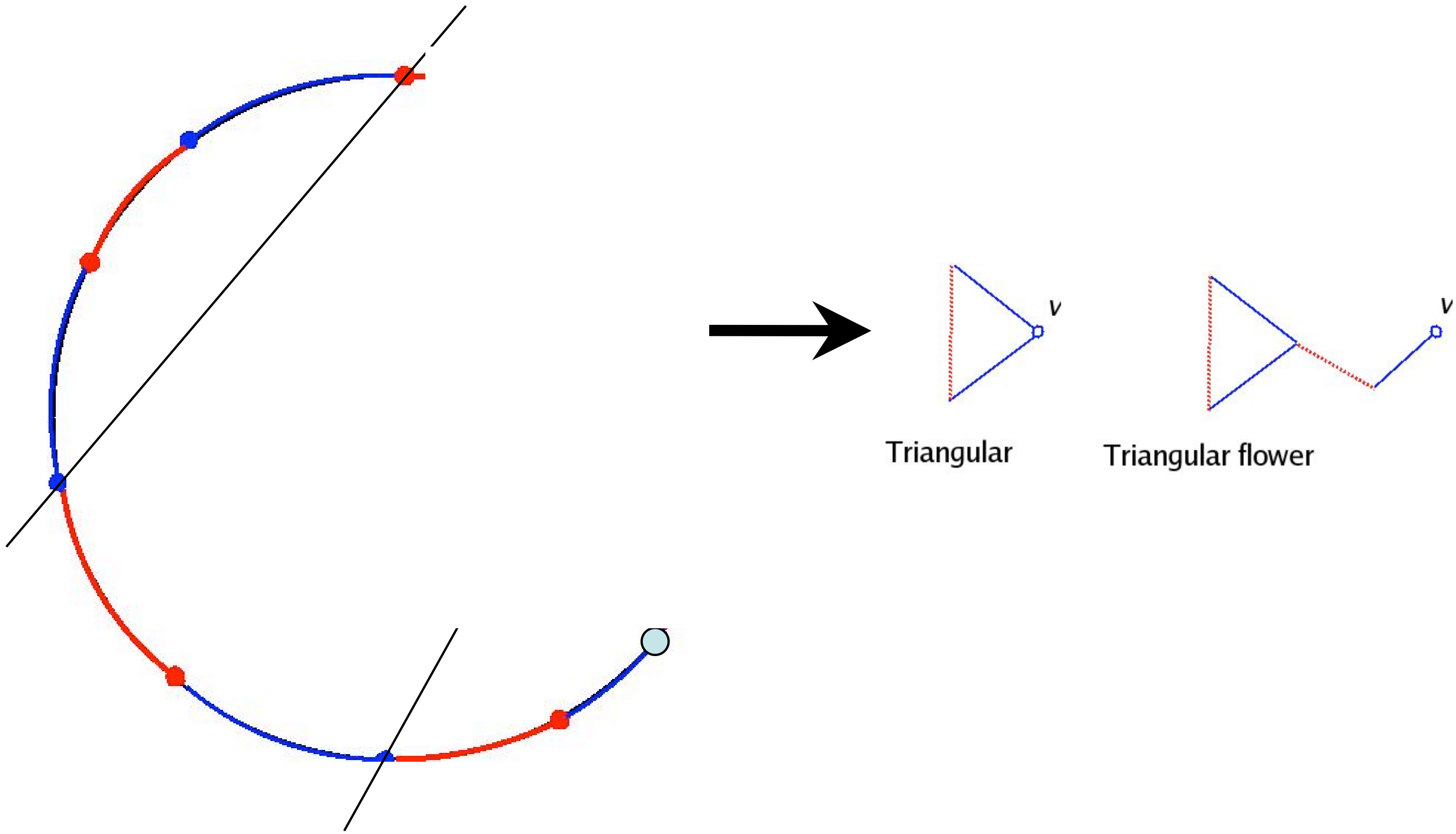

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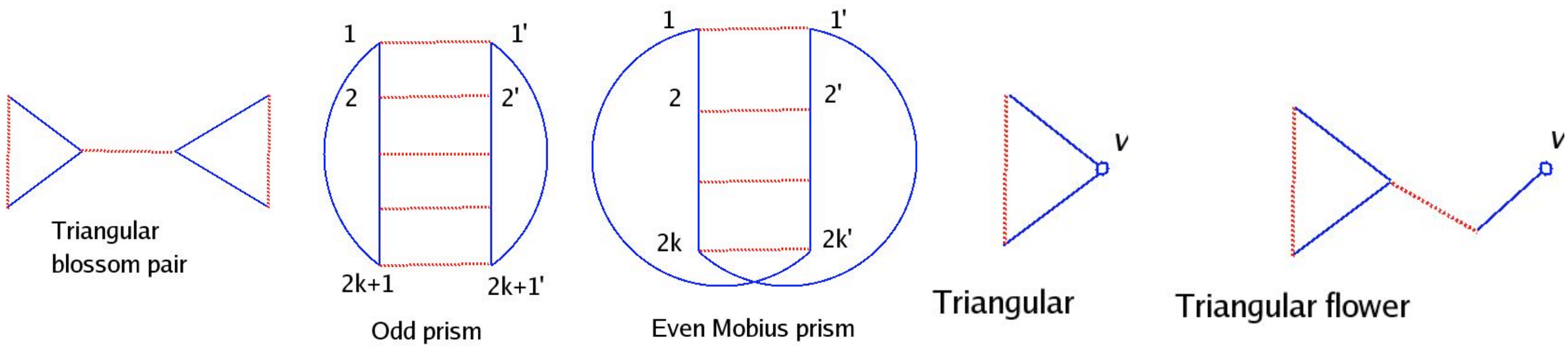

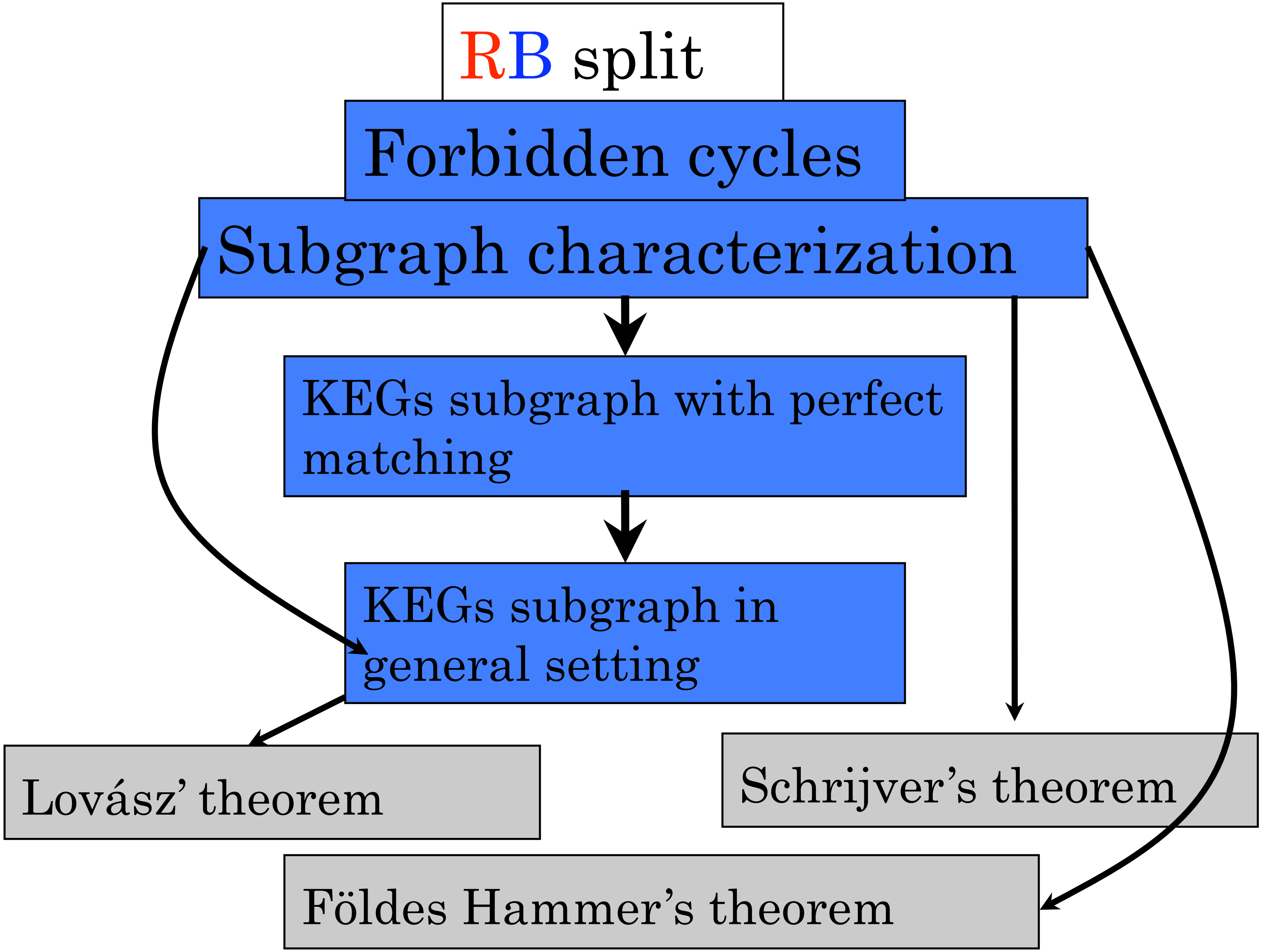
RB split
Forbidden cycles
Subgraph characterization
KEGs subgraph with perfect matching
KEGs subgraph in general setting
Lovász' theorem
Schrijver's theorem
Földes Hammer's theorem

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Thank you!!!