The Weighted Proportional Allocation Mechanism[1]

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Abstract–We consider a simple mechanism for allocation of resources to users that allows for discrimination of users by a revenue-maximizing provider. The mechanism is a natural extension of the well known proportional allocation by allocating proportional to weighted bids. The mechanism can be applied by providers whose resource constraints are described by general polyhedrons, which allows for complex resources such as, for example, networks of processors or links, data centers, and sponsored search.

We study a competitive system where users strategically choose their individual bids trying to maximize their payoffs and the provider aims at maximizing the revenue. We consider the revenue earned by the provider and the social welfare. Specifically, we find that the revenue under price anticipating users is lower bounded by $k/(k+1)$ times the revenue under standard third-degree price discrimination with a set of k users excluded. Under price anticipating users with linear utility functions, we find that the social welfare is at least $1/(1+2/\sqrt{3})\cdot 100 \approx 46\%$ of the maximum social welfare, and this bound is tight. We extend this result to a broad class of utility functions, which accomodates many utility functions found in literature, and to an oligopoly of multiple competing providers.

Our results indicate that the weighted proportional allocation simultaneously achieves competitive revenue and social welfare, under selfish users and selfish providers.

1 Introduction

Proportional allocation has been considered widely as a mechanism for allocation of resources; for example, for allocation of computer processors under Generalized Processor Sharing (Parekh and Gallager [28, 29]) or allocation of network links (Kelly [14]). The mechanism is typically applied for sharing of an infinitely divisible resource of a finite capacity by a set of users. Specifically, for an infinitely divisible resource of a positive capacity C, and a set of n users, each associated with a positive weight w_i, the allocation x_i to user i is set *proportional* the weight w_i as follows

$$x_i = \frac{w_i}{\sum_j w_j} C.$$

The mechanism is extended to systems of multiple individual resources by applying the proportional allocation mechanism per each individual resource.

The resource allocation under the proportional allocation mechanism can be interpreted as an auction where $w_1, \ldots, w_n$ are bids of individual users. A large body of literature focused on the *social welfare* of the proportional allocation under strategic user bidding, and to this end considered (worst-case) *efficiency*, defined as ratio of the social welfare at an equilibrium allocation and the maximum social welfare over the set of feasible allocations. Kelly [14] showed that for price taking users (i.e. users who submit their bids to maximize their individual payoffs assuming the price per unit resource is fixed), the system is fully efficient. In turn, Johari and Tsitsiklis [10] showed that efficiency is at least $3/4$ under price anticipating users, who select their bids to maximize their payoffs by accounting more fully for how the allocation depends on the submitted bids.

Proportional allocation mechanism is simple – bids are one-dimensional; the mechanism is amenable to a decentralized implementation through setting of the prices to marginal costs at individual resources and communicating to users the total price over their individually consumed resources; the mechanism is easy to understand by users.

We are interested in a mechanism that maximizes the revenue earned by the provider, equal to the sum of the submitted bids. The revenue maximization is an alternative objective to social welfare, well aligned with interests of a selfish provider. Ideally, one would hope for an allocation mechanism that simultaneously guarantees both large revenue and large social welfare. We would like that the mechanism equally applies to simple resources such as single processors or single network links as well as to more complex systems such as networks of processors or networks of communication links.

The Mechanism. We consider the following mechanism – we call weighted proportional allocation. For a resource with constraints specified by a polyhedron $\mathcal{P}$ in $\mathbb{R}_+^n$ ($\mathbb{R}_+ := [0,\infty)$), the allocation to user i is given by

$$x_i = \frac{w_i}{\sum_j w_j} C_i$$

where $\vec{w} = (w_1, \ldots, w_n)$ and $\vec{C} = (C_1, \ldots, C_n)$ are user's bids and discrimination weights such that $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{P}$. The payment by a user i is the bid w_i. Notice that the allocation x_i to user i is proportional to the product of the discrimination weight C_i and the bid w_i, and is in this sense a weighted proportional allocation.

The weighted proportional allocation mechanism allows for general resource constraints specified by polyhedrons. In particular, this accommodates infinitely divisible resources of finite capacities by $\mathcal{P} = \{\vec{x} \in \mathbb{R}_+^n : \sum_i x_i \leq C\}$, for $C > 0$; in this case, assuming full utililization, we have $x_i = \frac{w_i C_i}{\sum_j w_j C_j} C$. The proportional allocation is a special case with the discrimination weights set to a common value.

Furthermore, note that the weighted proportional allocation mechanism allows for price discrimination. Indeed, for a user i, the price per unit resource is $p_i = (\sum_j w_j)/C_i$ which may differ from one user to another due to the discrimination weights. The mechanism can be viewed as an auction that allows for price discrimination. Another interpretation is to regard the discrimination weights $C_1, \ldots, C_n$ as maximum allocations offered by the provider to users from which actual allocations then derive as portions of the maximum allocations proportional to users' bids. The weighted proportional allocation mechanism could be seen as an akin to generalized second price auction used in sponsored search [5, 37, 38, 18] for the following reasons. Therein, the advertisers (users) are ranked in decreasing order of the product of the bid and the click-through-rate (weight) associated to individual advertisers. Hence, the users are discriminated in a similar manner as with our weighted proportional allocation; in both settings, the purpose of the discrimination weights is the same – the revenue maximization.

Applications. The weighted proportional allocation mechanism could be used as a simple mechanism for service provisioning by providers; we indicate a few examples in the following. First, it could be used for allocation of data center resources to users; for example, for scheduling of multi-task jobs to machines in the context of data intensive computations. Typically, data centers are complex systems, consisting of many clusters of machines, and users have diverse processing and storage requirements. Second, it could be used for provisioning of network connectivity by network providers. In both two latter cases, detailed internal structure of the system would be known only by the provider, and only a simple interface would be available to users; some users may require more resources than others; the allocation mechanism would need to account for this diversity of requirements keeping a simple interface to users. Finally, it could be used in sponsored search for allocation of ad slots to advertisers. In this context, we interpret the allocation vector as click-through-rates over the ad slots, associated to particular assignment of ads to ad slots. The goal of the provider is to assign ad slots to a set of advertisers that maximizes the revenue over a convex closure of the observed allocation vectors; note that this approach accounts for externalities where the click-through-rate for a specific ad depends on the ads displayed along with this ad.

Summary of our Results. We consider the weighted proportional allocation in a competitive setting where both users and providers aim at selfishly optimizing their respective payoffs. The payoff of a user is the surplus, defined as the utility of a given allocation minus the payment. The payoff of the provider is the revenue, equal to the sum of the payments received from the users. We consider a weighted proportional allocation game where a provider announces the discrimination weights and then users strategically choose their bids. The provider anticipates the response from the users and choose the discrimination weights trying to maximize the revenue. We consider both price taking and price anticipating users, which are defined more precisely in Section 2.

We are interested in both the revenue and the social welfare under the weighted proportional allocation game. We consider both a monopoly scenario with a single provider and an oligopoly scenario with multiple providers. In the oligopoly scenario, each user can receive an allocation from any provider and is concerned only with the total allocation received across all the providers. Note that it is a priori rather unclear how efficient would be the weighted proportional allocation with respect to the revenue and the social welfare as both users and providers are selfish in optimizing their own objectives.

To the best of our knowledge, this paper would be a first attempt to consider the weighted proportional allocation for general polyhedron resource constraints. Our game-theoretic analysis provides insights into the revenue and the social welfare under the weighted proportional allocation; as a by-product, the analysis also tells about the standard proportional allocation in case of a strategic (revenue-maximizing) provider who may strategically misreport the capacity parameters that appear in the user payoff functions.

As standard, the revenue maximization requires from the provider to posses some information about utilities of individual users. In practice, this information would be readily available by providers whose service is repetitively used by users. In an implementation, the revenue maximization could be solved by an online optimization. In this paper, we consider the equilibrium points of the weighted proportional allocation game under complete information, i.e. providers have complete information about user's utilities.

Our results can be summarized in the following points:

- We find that under price anticipating users, the revenue of the weighted proportional allocation is at least $k/(k+1)$ times the revenue under standard third-degree price-discrimination with a set of k users excluded. The third-degree price-discrimination is revenue-equivalent to the weighted proportional allocation under price taking users. This revenue comparison result implies that in many cases with sufficiently large number of users, the revenue under the weighted proportional allocation would be near optimal.
- We show that for linear user utility functions, the social welfare is equal to the maximum social welfare under price taking users, and is at least $1/(1+2/\sqrt{3}) \approx 0.464$ times the maximum social welfare under price anticipating users; moreover, the latter bound is tight. This tells us, somewhat surprisingly, that the weighted proportional allocation guarantees the social welfare that is at least 46% of the maximum social welfare, even though the provider acts selfishly by maximizing the revenue and users act selfishly by maximizing their payoffs. This result is of independent interest as linear utility functions are rather commonly considered in literature as they well apply in some applications in practice. Furthermore, the result is used to generalize the efficiency lower bounds to a more general class of utility functions and multiple providers.
- We introduce a class of utility functions, we call δ-utility functions ($\delta \geq 0$), and show that this class of utility functions accommodates many families of utility functions found in literature. For example, a linear or truncated linear utility function belongs to this class as well as $\log(1+x)$, some polynomials and some families of utility functions commonly considered in the network resource allocation. We also show that a utility function from this class remains in the class by scaling with any positive constants and that sums of utility functions from this class remain in the class.
- We show that if the utility functions are δ-utility functions, then under price taking users the social welfare is at least $1/(1+\delta)$ times the maximum social welfare and under price anticipating users it is at least $1/(1+2/\sqrt{3}+\delta)$ times the maximum social welfare. Note that this result is established for an oligopoly of multiple competing providers. Hence, we establish that for the class of δ-utility functions, the social welfare is a constant factor of the maximum social welfare, even if, in addition, multiple providers compete in service provisioning to users.

Outline of the paper. In Section 2, we introduce the notation and assumptions. Section 3 provides the revenue comparison result (Theorem 1). In Section 4 we present our results on the social welfare for a monopoly of a single provider. We first present the result on the worst-case efficiency under linear utility functions (Theorem 2). We then define the class of δ-utility functions in Section 4.2. This is followed by a result on the worst-case efficiency for δ-utility functions (Theorem 3). Section 4.2.1 shows that many families of utility functions are δ-utility functions (Lemma 1 and Lemma 2), and finally, relates the class of δ-utility functions to the standard notion of the elasticity of demand. In Section 5, we consider an oligopoly of multiple competing providers and establish our worst-case efficiency bound for δ-utility functions (Theorem 5). We discuss the related work in Section 6. Finally, we conclude in Section 7. Some of the proofs are presented in Appendix.

2 Notation and Assumptions

We consider a system of n users competing for the resources of a single provider; we introduce the setting with multiple providers later in Section 5. Let x_i be the allocation to user i and let $\vec{x} = (x_1, \ldots, x_n)$ denote the allocation vector. Suppose that $U_i(x_i)$ is the utility of the allocation x_i to user i. The allocation vector $\vec{x}$ is feasible only if $\vec{x} \in \mathcal{P}$ where $\mathcal{P}$ is a set in R_+^n. We assume that $\mathcal{P}$ is a polyhedron, i.e. $\mathcal{P} = \{\vec{x} \in \mathbb{R}_+^n : A\vec{x} \leq \vec{b}\}$ for some matrix A and vector $\vec{b}$. Note that this allows for rather general resource constraints; for example, for any practical purposes, any convex set can well be approximated by a polyhedron. The special case of an infinitely divisible resource of a finite capacity C is accommodated by $\mathcal{P} = \{\vec{x} \in \mathrm{IR}_+^n : \sum_{i=1}^n x_i \leq C\}$. The general polyhedron constraints accommodate complex resources such as networks of processor or networks of communication links.

The weighted proportional allocation game is defined as follows. The provider announces a set of discrimination weights $\vec{C} = (C_1, \ldots, C_n)$ that are common knowledge, then users submit bids $\vec{w} = (w_1, \cdots, w_n)$. The allocation to each user i is given by

$$x_i = \frac{w_i}{\sum_j w_j} C_i. \tag{1}$$

We assume that each user i chooses a bid w_i that maximizes the surplus $U_i(x_i) - w_i$, respectively, under the price taking assumption (Section 2.1) and under the price anticipating assumption (Section 2.2). The provider anticipates the response by users and selects the discrimination weights $\vec{C}$ that maximize the revenue $R = \sum_j w_j$ such that the corresponding allocation vector $\vec{x}$ is feasible, i.e. $\vec{x} \in \mathcal{P}$.

2.1 Price Taking Users

Given a discrimination weight C_i and the sum of the bids $\sum_j w_j$, the price per unit resource for user i is $p_i = (\sum_j w_j)/C_i$. User i chooses a bid w_i that maximizes his surplus, taking the price p_i as fixed, i.e. solves

$$\text{USER:} \quad \text{maximize } U_i\left(\frac{w_i}{p_i}\right) - w_i \ \text{ over } w_i \geq 0.$$

It follows that $U_i'(x_i) = p_i$ for each user i, and hence the revenue is equal to $R(\vec{x}) = \sum_i U_i'(x_i) x_i$. The provider solves the following problem

$$\text{PROVIDER:} \quad \text{maximize } \sum_i U_i'(x_i) x_i \ \text{ over } \ \vec{x} \in \mathcal{P}. \tag{2}$$

A Nash equilibrium of the game under price taking users is a vector of discrimination weights $\vec{C}$ and a vector of user bids $\vec{w}$ such that for every user i,

$$C_i = \frac{R(\vec{x})}{U_i'(x_i)} \text{ and } w_i = U_i'(x_i) x_i$$

where $\vec{x} \in \mathcal{P}$ is a solution to (2).

Remark Indeed, if for each i, $U_i'(x)x$ is a concave function, then (2) is a maximization of a concave function over a convex set, hence a convex optimization problem.

Note that (2) is the same optimization problem as under standard third-degree price discrimination by a profit-maximizing monopoly [34, 36].

Remark The weighted proportional allocation under price taking users is revenue-equivalent to third-degree price discrimination.

2.2 Price Anticipating Users

Given a discrimination weight C_i and the sum of the bids $\sum_j w_j$, each user i selects a bid w_i that maximizes his surplus, i.e. solves

$$\text{USER:} \quad \text{maximize } U_i\left(\frac{w_i}{\sum_{j \neq i} w_j + w_i} C_i\right) - w_i \ \text{ over } w_i \geq 0. \tag{3}$$

The provider anticipates selection of the bids by users, and solves the following problem

$$\text{PROVIDER:} \quad \text{maximize } R(\vec{x}) \ \text{ over } \vec{x} \in \mathcal{P} \tag{4}$$

where $R(\vec{x})$ is given by

$$\sum_i \frac{U_i'(x_i)x_i}{U_i'(x_i)x_i + R(\vec{x})} = 1. \tag{5}$$

To see this, it suffices to show that for any given discrimination weights $\vec{C}$, the revenue is given by (5). Note that the objective function in (3) is concave in w_i, hence, at an optimum solution either $w_i = 0$ or the derivative of the objective function is zero. Setting the derivative to zero is equivalent to:

$$U_i'(x_i) \cdot \frac{\sum_{j \neq i} w_j}{(\sum_j w_j)^2} C_i - 1 = 0, \text{ for } x_i > 0.$$

It follows

$$U_i'(x_i) = \frac{(\sum_j w_j)^2}{C_i \sum_{j \neq i} w_j} = \frac{R^2}{C_i(R - w_i)} \tag{6}$$

where recall the revenue is equal to the sum of the payments by individual users, i.e. $R = \sum_j w_j$. Combining with $w_i = x_i R / C_i$ that follows from (1), we have

$$U_i'(x_i) = \frac{R}{C_i - x_i}. \tag{7}$$

From (7), we obtain

$$\frac{x_i}{C_i} = \frac{U_i'(x_i)x_i}{U_i'(x_i)x_i + R}.$$

Finally, combining with $\sum_i x_i / C_i = 1$ which follows from (1), we obtain (5). Note that all the formulas above are applied for the case $x_i > 0$ only; nevertheless, if $x_i = 0$, we have $U_i'(x_i)x_i = 0$, and therefore, the equation (5) holds for any optimum allocation vector $\vec{x}$.

A Nash equilibrium of the game under price anticipating users is a vector of discrimination weights $\vec{C}$ and a vector of user bids $\vec{w}$ such that for every user i,

$$C_i = x_i + \frac{R(\vec{x})}{U_i'(x_i)} \text{ and } w_i = \frac{R(\vec{x})}{U_i'(x_i)x_i + R(\vec{x})} U_i'(x_i)x_i$$

where $\vec{x} \in \mathcal{P}$ is a solution to (4).

We note the following proposition whose proof is deferred to Appendix 7.1.

Proposition 1 *Suppose that for each i, $U_i'(x)x$ is a concave function, then $R(\vec{x})$ is a concave function. Therefore, the provider problem (4) is a maximization of a concave function over a convex set, hence a convex optimization problem.*

Finally, we conclude this section with the following observations: (i) if $U_i'(x_i)x_i$ are not assumed to be concave, then (4) may have locally optimal solutions; (ii) if there is only one competing user, the revenue is 0; and (iii) for $n = 2$, the revenue can be expressed in the following explicit form, $R(\vec{x}) = \frac{1}{2}\sqrt{U_1'(x_1)x_1 U_2'(x_2)x_2}$.

3 Revenue

In this section, we compare the revenue of the weighted proportional allocation under price anticipating users with that of standard third-degree price discrimination [34, 36]. Recall that third-degree price discrimination is revenue-equivalent to weighted proportional allocation under price taking users.

Remark The optimum revenue of the weighted proportional allocation under price anticipating users is smaller then the optimum revenue of the weighted proportional allocation under price taking users.

The claims follows by noting from (5) that

$$R(\vec{x}) < \sum_i U_i'(x_i)x_i, \ \ \text{for all } \vec{x} \in \mathcal{P}.$$

For example, consider a system of n users, competing for a resource with the constraint $\sum_i x_i \le C$, for some $C > 0$. Suppose that users have a common utility function $U(x)$ such that $U'(x)x$ is concave. Then, the optimum revenue under price taking users is $n \max_{x\in[0,C/n]} U'(x)x$ while the optimum revenue under price anticipating users is smaller, equal to $(n-1)\max_{x\in[0,C/n]} U'(x)x$.

In the following, we provide a general revenue comparison result. Let R^*_{n-k} be the smallest optimum revenue under price taking users by excluding a set of k users. More formally,

$$R^*_{n-k} = \min_{S\subset\{1,\ldots,n\}:\ |S|=n-k} \ \max_{\vec{x}\in\mathcal{P}} \sum_{i\in S} U'(x_i)x_i.$$

Theorem 1 *Suppose that for each i, $U_i'(x)x$ is a concave function. Let R be the optimum revenue of weighted proportional allocation mechanism under price anticipating users, and let R^*_{n-k} be the smallest optimal revenue under price taking users by excluding a set of k users. Then, for $1 \le k < n$,*

$$R \ge \frac{k}{k+1} R^*_{n-k}.$$

Comments. The result implies that the revenue under price anticipating users is at least a factor $1/2$ of the optimum revenue achieved under price taking users by excluding a user whose exclusion decreases the revenue the most. For systems with large number of users, in many cases, the revenue would be near optimal.

Proof. From (5), it is straightforward to derive that

$$\sum_i U_i'(x_i)x_i - \max_j U_j'(x_j)x_j \le R(\vec{x}) < \sum_i U_i'(x_i)x_i, \text{ for all } \vec{x} \in \mathcal{P}.$$

Suppose that for each $1 \le k < n$, there exists $\vec{x} \in \mathcal{P}$ such that both of the following two conditions hold

(i) $\sum_i U_i'(x_i)x_i \ge R^*_{n-k}$;

(ii) $U_1'(x_1)x_1 = \cdots = U_{k+1}'(x_{k+1})x_{k+1} \ge \cdots \ge U_n'(x_n)x_n$,

where, without loss of generality, the users are enumerated such that $U_1'(x_1)x_1 \ge \cdots \ge U_n'(x_n)x_n$.

Under conditions i and ii, we have

$$R(\vec{x}) \ge \sum_i U_i'(x_i)x_i - \max_j U_j'(x_j)x_j \ge \frac{k}{k+1}\sum_i U_i'(x_i)x_i \ge \frac{k}{k+1}R^*_{n-k}$$

which is what we need to prove.

We show that conditions i and ii hold by induction over k. *Base step*: $k = 0$. In this case, condition i holds for $\vec{x}$ that is a solution to the provider problem under price taking users and condition ii is trivially true as there always exists a j such that $U_j'(x_j)x_j \ge U_i'(x_i)$ for each i. *Induction step*: suppose that there exists a vector $\vec{x} \in \mathcal{P}$ such that both condition i and condition ii hold for k. We then show that there exists a vector $\vec{x} \in \mathcal{P}$ such that conditions i and ii hold for $k+1$.

Let $\vec{x}$ be an allocation vector for which conditions i and ii hold for k. Note that $R^*_{n-k} \ge R^*_{n-(k+1)}$ as allowing to exclude a larger set of users cannot increase the smallest optimum revenue under price taking users. Combining with the induction hypothesis that condition i holds for k, we obtain $\sum_i U_i'(x_i)x_i \ge R^*_{n-(k+1)}$, i.e. condition i holds for $k+1$. It remains to show that condition ii holds for $k+1$.

In the following, without loss of generality, we assume that users are enumerated such that $U_1'(x_1)x_1 \ge \cdots \ge U_n'(x_n)x_n$. Let $\vec{y} \in \mathcal{P}$ be an optimum solution of the provider problem under price taking users under the constraint $y_1 = \ldots = y_{k+1} = 0$, i.e. with users $1, 2, \ldots, k+1$ excluded. Note that $\sum_i U_i'(y_i)y_i \ge R^*_{n-(k+1)}$.

Now, let us consider the vector $\vec{v}^t$ defined by

$$\vec{v}^t = (1-t)\cdot(U_1'(x_1)x_1,\ldots,U_n'(x_n)x_n) + t\cdot(U_1'(y_1)y_1,\ldots,U_n'(y_n)y_n),\ \text{ for } t\in[0,1].$$

Note that as t increases from 0, the $k+1$ largest coordinates of $\vec{v}^t$ decrease while all the other coordinates either increase or do not change. Thus, there exists $t^*\in[0,1]$ such that the largest $k+2$ coordinates of $\vec{v}^t$ are equal. Furthermore, as $\sum_i U_i'(x_i)x_i \geq R^*_{n-(k+1)}$ and $\sum_i U_i'(y_i)y_i \geq R^*_{n-(k+1)}$, we have that $\sum_i v_i^{t^*} \geq R^*_{n-(k+1)}$. Finally, since for each i, $U_i'(x_i)x_i$ is concave, there exists a vector $\vec{z}\in\mathcal{P}$ such that $(U_1'(z_1)z_1,\ldots,U_n'(z_n)z_n) = \vec{v}^{t^*}$. We showed that the vector $\vec{z}$ satisfies conditions i and ii for $k+1$ which completes the proof. ∎

4 Social Welfare

We consider social welfare under the weighted proportional allocation game. For an allocation vector $\vec{x}\in\mathcal{P}$, the social welfare is $\sum_j U_j(x_j)$. The worst-case efficiency is $\eta\in[0,1]$, if for any equilibrium allocation $\vec{x}$ of the weighted proportional allocation game, the following holds

$$\sum_i U_i(x_i) \geq \eta \sum_i U_i(y_i),\ \text{ for every } \vec{y}\in\mathcal{P},$$

i.e. the social welfare at any equilibrium of the game is at least the factor η of the maximum social welfare. This is a standard (worst-case) measure of efficiency, popularly referred to as the *price of anarchy* (Papadimitriou [27]), and considered by many under various assumptions (e.g. [10, 24, 7, 4, 9, 39, 12, 11, 1, 2]). Note that in the weighted proportional allocation game, any efficiency loss is a result of the selfish interests of users aiming at maximizing their payoffs, and the selfish interest of the provider aiming at maximizing the revenue.

We first establish a tight efficiency bound for the case of linear utility functions in Section 4.1. We then introduce a broad class of utility functions (we call δ-utility) in Section 4.2 and provide an efficiency bound for this class of utility functions. Finally, in Section 4.2.1, we show that many utility functions considered in literature are δ-utility functions.

4.1 Efficiency for Linear Utility Functions

We consider the worst-case efficiency for linear utility functions. The following result provides tight bounds on the worst-case efficiency under price taking and price anticipating users. The result is of independent interest as linear utility functions are rather commonly assumed in various contexts, e.g. sponsored search [5, 37, 38, 19]. We use this result in deriving efficiency bounds for a more general class of utility functions, for a single provider (Theorem 3) and multiple providers (Theorem 5).

Theorem 2 *Assume that for each i, the utility function $U_i(x)$ is strictly increasing and linear, i.e. for some $v_i>0$, $U_i(x)=v_ix$, for $x\geq 0$. Then,*

(i) under price taking users, the efficiency is 1*;*

(ii) under price anticipating users, the worst-case efficiency is $1/(1+2/\sqrt{3})$*. Moreover, this bound is tight.*

Proof is presented in Section 4.1.1.

Comments. The result establishes that under price anticipating users, the efficiency is at least $1/(1+2/\sqrt{3})$, which is approximately 46.41%. Furthermore, for every $\varepsilon>0$, there exist cases for which the efficiency is less or equal to $1/(1+2/\sqrt{3})+\varepsilon$. In comparison to the network resource allocation game considered by Johari and Tsitsiklis [10], the worst-case efficiency for the weighted proportional allocation game is lower. Indeed, [10] established the worst-case efficiency of $3/4\cdot 100 = 75\%$ under even more general user utility functions where users submit individual bids to resources applying the proportional allocation mechanism. In the setting of [10], the efficiency loss is

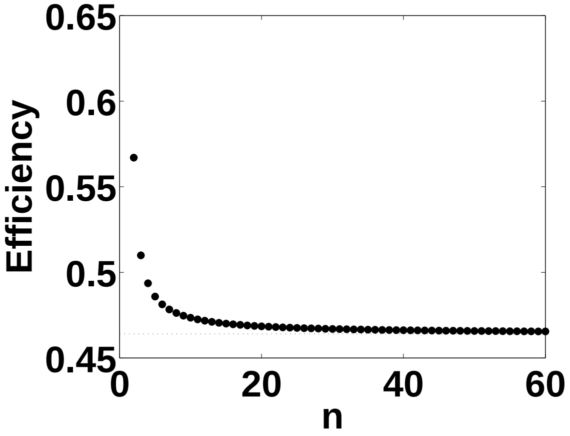

Figure 1: Worst-case efficiency versus the number of users n.

due to strategic user bidding while in our case it is due two factors: (i) strategic user bidding and (ii) revenue maximization by the provider; and this results in lower worst-case efficiency. The result stands in stark contrast to that of Yang and Hajek [7] who found that for the proportional allocation mechanism and users submitting scalar bids, the worst-case efficiency is 0 (their example is for linear utility functions). The theorem tells us that even under a revenue-maximizing provider, the social welfare is at least a constant factor of the maximum social welfare, where this constant is equal to one, under price taking users, and slightly smaller than one half, under price anticipating users. Hence, the weighted proportional allocation mechanism yields a competitive social welfare, under selfish users who aim at maximizing their payoffs and a selfish provider whose goal is to maximize his revenue. We will see later that this qualitative property holds also under more general utility functions, for a monopoly of single provider (Theorem 3) and an oligopoly of multiple providers (Theorem 5).

Remark In the proof of the theorem in Section 4.1.1, it is showed that the worst-case efficiency is achieved asymptotically as the number of users n tends to infinity, where there is a unique user with largest marginal utility v_i (say this is user 1) and all other users have identical marginal utilities equal to $(2-\sqrt{3})^2 v_1 \approx 0.0718 v_1$. The Nash equilibrium allocation is $\vec{x} = (x_1, x_2, \ldots, x_2)$ where

$$\begin{aligned} x_1 &= 1 - \frac{1}{\sqrt{3}} + o(1) \\ x_2 &= \frac{1}{\sqrt{3}} \frac{1}{n} + o(1/n). \end{aligned}$$

Hence, in the limit of many users, the user with the largest marginal utility is allocated 42.26% of the resource and the rest is equally shared by the remaining users.

Remark Since the tightness of the worst-case efficiency is showed as the number of users n goes to infinity (Section 4.1.1), one may legitimately ask how fast the worst-case efficiency converges to the limit with the number of users n. From Section 4.1.1, it is not difficult to note that the worst-case efficiency is $x_1 + (1-x_1)\frac{v_2}{v_1}$ where v_2/v_1 and x_1 are given by (17) and (18), respectively. We present the worst-case efficiency versus n in Figure 1; note that the worst-case efficiency is approximately 56% for two users and it gets near to the asymptote $1/(1+2/\sqrt{3})$ already for tens of users. In fact, one can show that

$$\text{worst-case efficiency} = \frac{1}{1+\frac{2}{\sqrt{3}}} \left(1 + \frac{2(2-\sqrt{3})}{3} \frac{1}{n}\right) + o(1/n).$$

Finally, we show that the larger the competitiveness among the users, the larger the efficiency. This is made precise in the following proposition whose proof is deferred to Appendix 7.3.

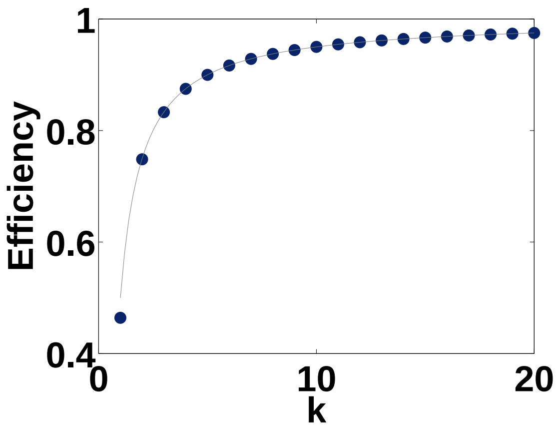

Figure 2: Worst-case efficiency versus k. The solid line is the asymptote $1-\frac{1}{2k}+o(1/k)$.

Proposition 2 *Assume the same setting as in Theorem 2 but in addition, assume* $v_1=\cdots=v_k\geq v_{k+1}\geq v_n$. *Then, under price anticipating users, the efficiency is at least* $1-\frac{1}{2k}+o(1/k)$.

Remark In Figure 2, we show numerical results for the worst-case efficiency under the assumption that at least k users have largest marginal utilities, along with the asymptote in Proposition 2; note that the efficiency is almost 75% for $k=2$, larger than 80% for $k=3$, and almost 90% for $k=5$; the asymptote of the proposition is a good approximation for every $k\geq 2$.

4.1.1 Proof of Theorem 2

Item i easily follows as under linear utility functions, $U_i'(x)x=U_i(x)$, for $x\geq 0$, for every i, hence, the revenue maximization corresponds to social welfare maximization.

In the remainder we show item ii. Let $U_i(x)=v_ix$, $v_i>0$, for each user i. Let $R(\vec{x})$ be the function given by (5). Let R^* be the optimum revenue, i.e. $R^*=\max\{R(\vec{x}):\ \vec{x}\in\mathcal{P}\}$. We note the following:

Claim 1 *The set* $\mathcal{L}_\mu:=\{\vec{x}\in\mathbb{R}_+^n: R(\vec{x})\geq\mu\}$ *is convex, for every* $\mu\in[0,R^*]$.

The claim indeed holds as in Proposition 1 we showed that $R(\vec{x})$ is a concave function, and thus $\mathcal{L}_\mu$ is a level-set of a concave function $R(\vec{x})$ which is convex [30][Theorem 4.6].

Since for every $\vec{x}\in\mathcal{P}$, $R(\vec{x})\leq R^*$, the two convex sets $\mathcal{L}_{R^*}$ and $\mathcal{P}$ do not have common interior points. Let H be a hyperplane that weakly separates these two sets. This hyperplane can be written as

$$\sum_i\gamma_ix_i=1,\ \text{with}\ \gamma_i\geq 0\ \text{for each}\ i. \tag{8}$$

In a slightly different game, where the provider has the feasible set $\mathcal{Q}=\{\vec{x}\in\mathbb{R}_+^n:\sum_i\gamma_ix_i\leq 1\}$, the allocation that maximizes the revenue over $\mathcal{Q}$ is the same as in the original game. Since $\mathcal{P}\subset\mathcal{Q}$, the optimal social welfare of the new game is at least the social welfare of the original game. Therefore, it is enough to prove a lower bound on the efficiency for the class of games where the provider has the feasible set $\mathcal{Q}$.

Taking the partial derivative with respect to x_j on both sides in (5), we obtain

$$\frac{[U_j'(x_j)x_j]'}{(U_j'(x_j)x_j+R)^2}=\frac{\frac{\partial}{\partial x_j}R}{R}\cdot\sum_i\frac{U_i'(x_i)x_i}{(U_i'(x_i)x_i+R)^2}.$$

For any optimum allocation vector $\vec{x}$, we have either $x_j = 0$ or $\frac{\partial}{\partial x_j} R = \lambda \gamma_j$ where $\lambda \geq 0$ is the Lagrange multiplier [30] associated to the constraint $\sum_i \gamma_i x_i \leq 1$. It follows that

$$\text{either } x_i = 0 \text{ or } \frac{\frac{1}{\gamma_i}[U_i'(x_i)x_i]'}{(U_i'(x_i)x_i + R^*)^2} = p > 0. \tag{9}$$

In the following, it is convenient to use the following notation

$$a_i = \frac{v_i}{\gamma_i} \text{ and } z_i = \gamma_i x_i,\ y_i = \frac{a_i z_i}{a_i z_i + R^*},\ \text{for each user } i,$$

where $\vec{x}$ is a Nash equilibrium allocation vector. Without loss of generality, assume that

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

The equalities (5) and (8) can now be written as

$$\begin{aligned} \sum_i y_i &= 1 && (10) \\ \sum_i z_i &= 1. && (11) \end{aligned}$$

For linear utility functions, from (9),

$$\text{either } z_i = 0 \text{ or } \frac{a_i}{(a_i z_i + R^*)^2} = p > 0. \tag{12}$$

It is straightforward to note that the following holds

$$a_i z_i = R^* \frac{y_i}{1 - y_i} \tag{13}$$

and that (12) is the same as

$$\text{either } y_i = 0 \text{ or } \frac{a_i(1-y_i)^2}{R^{*2}} = p > 0. \tag{14}$$

On the one hand, from (13) and (10), we have that the social welfare at a Nash equilibrium satisfies

$$\sum_i a_i z_i = R^* \sum_i \frac{y_i}{1-y_i} \geq R^* \left(\frac{y_1}{1-y_1} + \sum_{i \geq 2} y_i \right) = R^* \left(\frac{y_1}{1-y_1} + (1-y_1) \right) = R^* \frac{y_1^2 - y_1 + 1}{1-y_1}. \tag{15}$$

On the other hand, using (11) and (13), the maximum social welfare is

$$a_1 = a_1 \left(\sum_i z_i \right) = a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)}.$$

From (14), $a_i(1-y_i)^2 = a_1(1-y_1)^2$ whenever $y_1, y_i > 0$, hence

$$a_1 R^* \sum_i \frac{y_i}{a_i(1-y_i)} = \frac{R^*}{(1-y_1)^2} \sum_i y_i(1-y_i).$$

Hence, for the maximum social welfare

$$\begin{aligned} \frac{R^*}{(1-y_1)^2} \sum_i y_i(1-y_i) &\leq \frac{R^*}{(1-y_1)^2} \left(y_1(1-y_1) + \sum_{i>1} y_i \right) \\ &= \frac{R^*}{(1-y_1)^2} (y_1(1-y_1) + 1 - y_1) \\ &= R^* \frac{1-y_1^2}{(1-y_1)^2} && (16) \end{aligned}$$

where the second equality follows from (10).

From (15) and (16), the efficiency is at least

$$\frac{R^* \frac{y_1^2 - y_1 + 1}{1 - y_1}}{R^* \frac{1 - y_1^2}{(1 - y_1)^2}} = \frac{y_1^2 - y_1 + 1}{y_1 + 1} \geq \frac{1}{1 + \frac{2}{\sqrt{3}}}$$

where the inequality follows from the following claim

Claim 2 *The minimum of the function* $f(y) = \frac{y^2 - y + 1}{y + 1}$ *over* $[0, 1]$ *is equal to* $2\sqrt{3} - 3 = 1/(1 + 2/\sqrt{3})$.

whose proof is given in Appendix 7.2. It remains only to prove that the asserted efficiency bound is tight which we do in the following.

Tightness. Consider the following case: $v_1 > v_2 = \cdots = v_n$ with

$$\frac{v_2}{v_1} = \left(\frac{1 - y_1}{1 - y_2} \right)^2 \tag{17}$$

where $y_1 = \sqrt{3} - 1$, $y_2 = (2 - \sqrt{3})/(n - 1)$, and let $\gamma_i = 1$ for each user i, i.e. $\sum_i x_i = 1$.

For this choice, note that (14) holds. Furthermore, from (13) we have that $\vec{x} = (x_1, x_2, \dots, x_2)$ is a Nash equilibrium allocation vector such that

$$\frac{v_2 x_2}{v_1 x_1} = \frac{y_2(1 - y_1)}{y_1(1 - y_2)}.$$

It follows

$$\frac{x_2}{x_1} = \frac{y_2(1 - y_2)}{y_1(1 - y_1)}.$$

Combining with $x_1 + (n - 1)x_2 = 1$, we have

$$x_1 = \frac{1}{1 + (n - 1)\frac{y_2(1 - y_2)}{y_1(1 - y_1)}} \tag{18}$$

and $x_2 = (1 - x_1)/(n - 1)$. Evaluating the ratio of the social welfare at the Nash equilibrium and the maximum social welfare v_1, we obtain $1/(1 + 2/\sqrt{3})$ asymptotically as n tends to infinity. This completes the proof.

4.2 Efficiency for a General Class of Utility Functions

In the previous section, we showed that if the user utility functions are linear, then the efficiency is bounded by a constant that is independent of the number of users. However, in literature, user utility functions are often assumed to be non-negative, non-decreasing concave functions, which is more general. We first argue that for this more general class of utility functions, there exist cases for which the efficiency can be arbitrarily small under a revenue-maximizing provider. We then define a broad class of utility functions which accommodates many utility functions used in the literature. For this class of utility functions, we prove that the efficiency is bounded by a constant for a monopoly of a single provider and an oligopoly of multiple providers (Section 5).

A bad example. We show an example where user utility functions are non-negative, non-decreasing concave functions, for which the efficiency is 0. The provider optimizes $R(\vec{x})$ given by (5). $R(\vec{x})$ is an increasing function in $U_i'(x_i)x_i$, therefore, for general concave utility functions, it might happen that there is a Nash equilibrium allocation $\vec{x}^* \in \mathcal{P}$ such that $U_i'(x_i^*)x_i^* = \max_{x \in [0,\infty)} U_i'(x)x$ and the provider will have incentive to under report the availability of the resource in order to maximize the revenue. This is the main, common cause of efficiency loss under revenue-maximizing providers.

Consider a symmetric case of 2 users with $U_i(x) = U(x)$ for $i = 1, 2$, and a provider with the resource constraint $x_1 + x_2 \leq C$, for $C > 0$. Suppose $U(x)$ is a non-negative, non-decreasing concave function specified by

$$U'(x) = \begin{cases} 3-2x & 0 \le x \le 1 \\ \frac{1}{x} & x \ge 1. \end{cases}$$

We then have that the provider maximizes $R(x) = U'(x)x$ over $0 \le x \le C/2$, with the maximizing point $x = 3/4$. The ratio of the social welfare at the Nash equilibrium and the maximum social welfare is $U(3/4)/U(C/2)$. Letting C go to infinity, we obtain the limit efficiency $\lim_{C\to\infty} U(3/4)/U(C/2) = 0$.

The preceding example shows that we need to restrict the class of utility functions to a subset of non-negative, non-decreasing concave functions, in order to guarantee a constant efficiency for any number of users. To this end, we introduce the class of δ-utility functions in the following definition.

Definition 1 *Let $U(x)$ be a non-negative, non-decreasing, and concave utility function and let $x_0 \ge 0$ be the value maximizing $U'(x)x$. We call $U(x)$, δ-utility, if, in addition, the following two conditions hold*

(i) $U'(x)x$ is a concave function over $[0,x_0]$, and

(ii) there exists $\delta \in [0,\infty)$, such that for every $a \in [0,x_0]$,

$$U(b) \le \delta U(a) + [U'(a)a]'b, \text{ for all } b \ge 0. \tag{19}$$

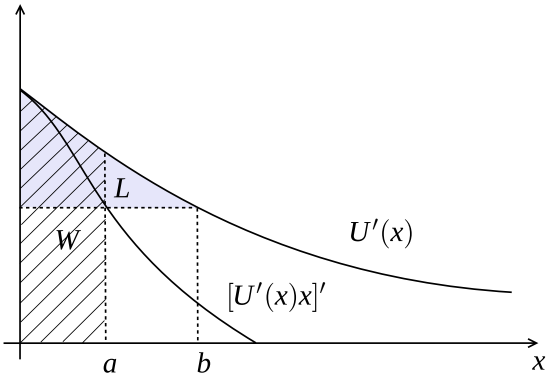

Figure 3: A δ-utility function: $\frac{L}{W} \le \delta$, where L is the shaded and W is the hatched area.

Condition (19) means that the utility function $U(x)$ is upper bounded by *affine* functions $c + dx$ where $c = \delta U(a)$ and $d = [U'(a)a]'$, for every $a \in [0,x_0]$. Note that condition (19) is equivalent to requiring that for every $a \in [0,x_0]$,

$$U(b) - U'(b)b \le \delta U(a)$$

where b is given by $U'(b) = [U'(a)a]'$. We show a geometric interpretation of condition (19) in in Figure 3 where it corresponds to $L/W \le \delta$, for very $a \in [0,x_0]$, where L and W are the areas indicated in Figure 3.

We next briefly discuss why the above class of utility functions is rather natural, postponing to show that the class accommodates many utility functions found in literature to Section 4.2.1.

Relation to the efficiency of monopoly pricing. The dependency of the efficiency on the properties of the utility functions holds in general, not is not particular to our mechanism. Consider the classical case of monopoly pricing [36] – a single seller sells to buyers with utility function $U(x)$ and there is a constant marginal production cost $c > 0$. The seller optimizes the price p so as to maximize the profit. For given price p, buyers choose the quantity x that maximizes $U(x) - px$, so that $U'(x) = p$. The sellers finds the quantity x^m that maximizes $p(x)x - cx = U'(x)x - cx$, thus $[U'(x^m)x^m]' = c$. The social welfare is maximum at a quantity x^s that maximizes $U(x) - cx$, thus $U'(x^s) = c$. It follows that the efficiency is $(U(x^m) - cx^m)/(U(x^s) - cx^s)$, which in the geometric interpretation in Figure 6 corresponds to the ratio W'/L. Note that if the utility function is such that for some $\gamma \in [0,1]$, the efficiency of the monopoly pricing is greater or equal to γ for any marginal cost $c > 0$, then the utility function is a $\frac{1}{\gamma}$-utility.

We now state the main result of this section:

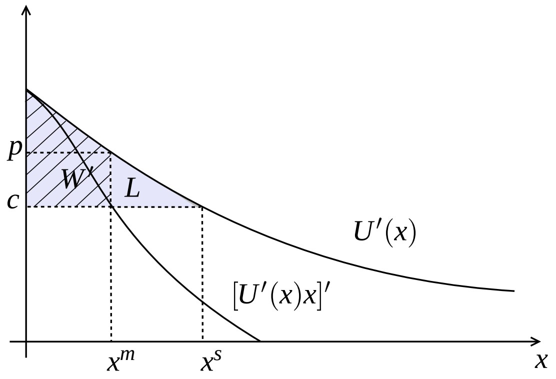

Figure 4: Efficiency for monopoly pricing: $\frac{W'}{L}$.

Theorem 3 *Assume that for each i, the utility function $U_i(x)$ is a δ-utility. Then,*

(i) under price taking users, the efficiency is at least $1/(1+\delta)$;

(ii) under price anticipating users, the efficiency is at least $1/(1+2/\sqrt{3}+\delta)$.

Proof. This is a corollary of Theorem 5 which holds for a more general setting of $m \geq 1$ competing providers. ■

Note that for $\delta = 0$, we recover the result of Theorem 2. The result under item ii shows that for the price anticipating users with utility functions in the class of δ-utility functions, the efficiency is at least $1/(1+2/\sqrt{3}+\delta) \approx 1/(2.16+\delta)$. This shows that the weighted proportional allocation yields a competitive social welfare under a broad class of δ-utility functions.

In the following section, we exhibit that some common families of utility functions belong to the class of δ-utility functions and for these families of utility functions compute the values of δ.

4.2.1 δ-utility Functions

The following lemma shows that many utility functions found in literature are δ-utilities.

Lemma 1 *We have the following properties:*

(i) $U(x) = \alpha x$, for $\alpha > 0$, or a truncated linear function[1], is a 0-utility;

(ii) $U(x)$ such that $U'(x)$ is a concave function is a 2-utility;

(iii) $U(x) = \log\left(\frac{c+x}{c}\right)$, for $c > 0$, is a 2-utility;

(iv) $U(x) = (c+x)^\alpha$ for $c \geq 0$ and $0 \leq \alpha \leq 1$ is a $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$-utility for $0 \leq \alpha \leq \frac{1}{2}$, and a 1-utility for $\frac{1}{2} < \alpha \leq 1$; simpler but weaker, it is a $\frac{e}{2}$-utility for $0 < \alpha \leq 1$.

(v) $U(x) = \frac{w^\alpha}{1-\alpha}(c+x)^{1-\alpha}$, for $\alpha \in [0,1) \cup (1,\infty)$, and $U(x) = w\log(c+x)$, for $\alpha = 1$, with $w > 0$ and any $c \geq 0$, is a 1-utility for $0 \leq \alpha \leq \frac{1}{2}$ and a $\alpha(1-\alpha)^{-\frac{1-\alpha}{\alpha}}$-utility for $\frac{1}{2} < \alpha < 1$; simpler but weaker, it is a $\frac{e}{2}$-utility for $0 \leq \alpha < 1$.

(vi) $U(x) = \alpha \cdot \arctan\left(\frac{x}{\alpha}\right)$, for $\alpha > 0$, is a 2-utility.

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[1] That is, $U(x) = \min\{\alpha x, y\}$, for every $x \geq 0$, for some $y > 0$.

The linear functions under item i are rather commonplace in economics and other literature and apply well to situations where users derive constant marginal rewards, e.g. a fixed reward per click on an ad [18]; truncated linear utility functions were considered representative of real-time traffic requirements in communication network scenarios [31] and were considered in the context of price competition games [1, 2]. Concave marginal utilities (item ii) were considered in [9]. The logarithmic utility function under item iii was considered by several authors, for instance, as a model of a communication network connection in [3] and [31]; it can be seen as an approximation for proportional-fair allocations [15]. Polynomial utility functions (item iv) were used in a model of trade [17]. The family of utility functions under item v is known as α-fair [15, 22] and was used widely in the context of network resource sharing. Finally, the utility function under item vi was derived by Kelly [13] as a characterization of a TCP-like connection.

We have the following result whose proof can be found in Appendix 7.5.

Lemma 2 *If f and g are δ-utilities, then so are:*

(i) $c \cdot f$, *for* $c > 0$;

(ii) truncated f

(iii) $f + g$.

Relation to the elasticity of demand. It is natural to consider how the definition of δ-utility compares to existing characterizations of utility functions. A standard measure is the so-called *elasticity of demand* which is used to characterize the third-degree price discrimination [36, 34]. Let $p(x) = U'(x)$ where $p(x)$ is interpreted as the price at the output x, and the inverse function $x(p)$ is known as the demand function. The elasticity of demand at the output x is defined by $-(dx/x)/(dp/p)$. Note that this corresponds to $U'(x)/(-U''(x)x)$. Now, instead of considering the elasticity of demand at particular values of the output x, we consider the following uniform bound, let $\varepsilon > 0$ be such that

$$\frac{U'(x)}{-U''(x)x} \geq \varepsilon, \quad \text{for all } x \geq 0. \tag{20}$$

Note that in Figure 3, the left-hand side corresponds to the ratio of the length of the line segment $[(a,0),(a,U'(a))]$ and the length of the line segment $[(a,[U'(a)a]'),(a,U'(a))]$. Intuitively, we want this ratio to be large as we want the area L to be small relative to the area W. This indeed conforms to the fact that linear utility functions have infinite elasticity of demand, and by Lemma 1 we know that linear utility functions are 0-utilities.

The following lemma provides a relation between the elasticity of demand and δ-utility.

Lemma 3 *Suppose $U(x)$ is a non-negative utility function such that $U'(x)$ is non-increasing and concave and (20) holds. Then, $U(x)$ is a $\frac{2}{\varepsilon}$-utility function.*

Proof. For a concave function $U'(x)$, the area L in Figure 3 is less than or equal to the area of the triangle defined by the intersection of the lines $x = 0$, $y = [U'(a)a]'$, and the tangent to the function $U'(x)$ at $x = a$. The area of this triangle is $-2U''(a)a^2$. Hence,

$$\frac{L}{W} \leq \frac{-2U''(a)a^2}{U(a)} = \frac{-2U''(a)a}{U'(a)}\frac{U'(a)a}{U(a)} \leq \frac{-2U''(a)a}{U'(a)} \leq \frac{2}{\varepsilon}$$

where the first inequality follows as $U'(a)a \leq U(a)$ holds for any non-negative concave utility function $U(a)$ (which holds as $U'(a)$ is assumed to be non-increasing) and the last inequality is by (20). ■

The following example shows that there exist utility functions for which strictly positive (and even large) elasticity does not imply that the utility function is a δ-utility. Indeed, such utility functions do not verify the assumptions of Lemma 3.

Example. For $c, d > 0$, and $0 < x_0 \leq c/d$, let

$$U'(x) = \begin{cases} c - \frac{d}{2}x & 0 \leq x \leq x_0 \\ c - dx_0 + \frac{dx_0^2}{2x} & x > x_0. \end{cases}$$

Note that $[U'(x)x]' = c - dx$, for $0 \leq x \leq x_0$, and $[U'(x)x]' = c - dx_0$, otherwise. Note that $U'(x)$ is a convex function and thus the assumptions of Lemma 3 are not met. It is not difficult to find that

$$\varepsilon = \frac{2c}{dx_0} - 1 > 0.$$

For a finite $a > x_0$, we have that the area W (see Figure 3) is equal to $U(a) = \frac{3d}{4}x_0^2 + (c - dx_0)a + \frac{dx_0^2}{2}\log\frac{a}{x_0}$, thus $0 < W < \infty$. On the other hand, for the area L we have $L = (U(a) - [U'(a)a]'a) + \frac{dx_0^2}{2}\int_a^\infty \frac{dx}{x} = \infty$, thus $U(x)$ is not a δ-utility.

5 Multiple Providers

In this section, we consider an oligopoly of multiple competing providers where each provider allocates resources according to the weighted proportional allocation. We assume that each user can receive resources from any provider and is concerned only with the total allocation received over all providers. More concretely, let x_i^k denote the allocation to user i by provider k. We assume that for each user i, the utility of the allocation vector $(x_i^1,\ i = 1, \ldots, n,\ k = 1, \ldots, m)$ is a function $U_i(x_i)$ where $x_i = \sum_k x_i^k$ denote the total allocation to user i over all providers. In the remainder we denote with $x_i^{-k} = x_i - x_i^k$ the total allocation to user i over all providers except provider k, and use the following vector notation $\vec{x}^k = (x_1^k, \ldots, x_n^k)$ and $\vec{x}^{-k} = (x_1^{-k}, \ldots, x_n^{-k})$.

Each user i chooses bids $\vec{w}_i = (w_i^1, \ldots, w_i^m)$ that maximize

$$\text{USER:}\ \ \text{maximize}\ U_i(x_i) - \sum_k w_i^k \tag{21}$$

over $\vec{w}_i \in \mathbb{R}_+^n$ subject to either price taking or price anticipating which we discuss in the following. In either case, we have

$$x_i = \sum_k \frac{w_i^k}{p_i^k}$$

where $p_i^k := (\sum_j w_j^k)/C_i^k$ is the price per unit resource for user i charged by provider k and C_i^k is the discrimination weight for user i by provider k.

Price taking users. Under price taking users, the maximization in (21) is performed over $\vec{w}_i \in \mathbb{R}_+^n$ while taking the prices p_i^k as fixed. It is easy to note that for an optimal allocation $(x_i^1, \ldots, x_i^m)$, either $x_i^k = 0$ or $U_i'(x_i) = p_i^k$. Therefore, the revenue by provider k is equal to $R^k(\vec{x}^k, \vec{x}^{-k}) = \sum_i U_i'(x_i^{-k} + x_i^k)x_i^k$.

A Nash equilibrium of the game under price taking users is a vector of discrimination weights $(C_i^k, i = 1, \ldots, n, k = 1, \ldots, m)$ and a vector of user bids $(w_i^k, i = 1, \ldots, n, k = 1, \ldots, m)$ such that

$$C_i^k = \frac{R^k(\vec{x}^k, \vec{x}^{-k})}{U_j'(x_i)} \text{ and } w_i^k = U_i'(x_i)x_i^k$$

where $(\vec{x}^1, \ldots, \vec{x}^m)$ is a set of allocation vectors $(\vec{x}^1, \ldots, \vec{x}^m) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$ such that for every provider k,

$$R^k(\vec{x}^k, \vec{x}^{-k}) \geq R^k(\vec{y}^k, \vec{x}^{-k}),\ \text{for every}\ \vec{y}^k \in \mathcal{P}_k.$$

Price anticipating users. Following same arguments as for the case of a single provider in Section 2.2, we have that for any vector of discrimination weights $(C_i^k, i = 1,\dots,n, k = 1,\dots,m)$, the price anticipating users select bids so that the following relations hold for every i and every k,

$$C_i^k = x_i^k + \frac{R^k(\vec{x}^k,\vec{x}^{-k})}{U_i'(x_i)} \text{ and } w_i^k = \frac{R^k(\vec{x}^k,\vec{x}^{-k})}{U_i'(x_i)x_i^k + R^k(\vec{x}^k,\vec{x}^{-k})} U_i'(x_i)x_i^k \tag{22}$$

where $R^k(\vec{x}^k,\vec{x}^{-k}) = \sum_i w_i^k$ is the revenue of provider k given by

$$\sum_{i=1}^{n} \frac{U_i'(x_i^{-k} + x_i^k)x_i^k}{U_i'(x_i^{-k} + x_i^k)x_i^k + R^k(\vec{x}^k,\vec{x}^{-k})} = 1. \tag{23}$$

A Nash equilibrium of the game under price anticipating users is a vector of discrimination weights $(C_i^k, i = 1,\dots,n, k = 1,\dots,m)$ and a vector of user bids $(w_i^k, i = 1,\dots,n, k = 1,\dots,m)$ that satisfy (22) for every i and every k, where $(\vec{x}^1,\dots,\vec{x}^m)$ is a set of allocation vectors $(\vec{x}^1,\dots,\vec{x}^m) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$ such that for every provider k,

$$R^k(\vec{x}^k,\vec{x}^{-k}) \geq R^k(\vec{y}^k,\vec{x}^{-k}), \text{ for every } \vec{y}^k \in \mathcal{P}_k.$$

We first show that there exists a Nash equilibrium for our multi-provider weighted proportional allocation game.

Theorem 4 *Assume that for each user i and every* $a \geq 0$, $U_i'(x+a)x$ *is a concave function. Then, there exists a Nash equilibrium for our multiple-provider weighted proportional allocation game.*

Proof. For both price taking users and price anticipating users, a Nash equilibrium is determined by a set of allocation vectors $(\vec{x}^1,\dots,\vec{x}^m) \in \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$. Consider the conventional best-response function

$$F : \mathcal{P}_1 \times \cdots \times \mathcal{P}_m \to \mathcal{P}_1 \times \cdots \times \mathcal{P}_m$$

such that $(\vec{y}^1,\dots,\vec{y}^m) = F(\vec{x}^1,\dots,\vec{x}^m)$, where $\vec{y}^k$ is the allocation vector that maximizes the revenue for provider k, assuming other providers do not change their allocations. Since the revenue is a solution of a convex optimization problem (immediate under price taking users; follows by same arguments as for Proposition 1 under price anticipating users), this mapping is continuous and thus by the fixed-point theorem [16], there exists an allocation vector where no provider k can increase his revenue by changing the allocation vector $\vec{x}^k$, which is a Nash equilibrium. ■

We next prove a bound on the efficiency at any equilibrium allocation of the multi-provider weighted proportional allocation game. In order to preclude some trivial cases, we admit the following assumption. Each provider k allocates no resources to any user i for which $U_i'(x_i) = 0$, for all $\vec{x}^k \in \mathcal{P}_k$, i.e. in the given case $x_i^k = 0$. This ensures

$$U_i'(x_i) > 0 \text{ whenever } x_i > 0. \tag{24}$$

Theorem 5 *Assume that for each user i and every* $a \geq 0$, $U_i(a+x)$ *is a non-constant* δ*-utility. Then,*

(i) under price taking users, the efficiency is at least $1/(1+\delta)$*;*

(ii) under price anticipating users, the efficiency is at least $1/(1+2/\sqrt{3}+\delta)$.

Comments. The result shows that the efficiency of the weighted proportional allocation in the multi-provider setting is at least $1/(1+\delta)$ and $1/(2.16+\delta)$ under price taking and price anticipating users, respectively. The result is tight in the sense that for $\delta = 0$ (which holds for linear utility functions), it recovers the same efficiency bounds as those in Theorem 2, which were showed to be tight. It establishes that for the class of δ-utility functions, the social welfare is a constant factor of the maximum social welfare, which depends on δ and is independent of the number of users and the number of providers.

Proof. The key idea of the proof is to bound the social welfare by an affine function which allows separating the maximization over $(x_1, \ldots, x_n) \in \sum_k \mathcal{P}_k$ to maximizations over the sets $\mathcal{P}_k$, where $\sum_k \mathcal{P}_k := \{\vec{z}^1 + \cdots + \vec{z}^m : \vec{z}^k \in \mathcal{P}_k, k = 1, \ldots, m\}$ is the Minkowski sum of the sets $\mathcal{P}_k$.

We first consider the sub-game for provider k, assuming that $\vec{x}^j$ is fixed, for every $j \neq k$. For this sub-game, let $\vec{x}^k$ and R^k denote a Nash-equilibrium allocation and the revenue for provider k, respectively. Let $\sum_i \gamma_i^k x_i^k = 1$ be the hyperplane that separates the sets $\mathcal{L}^k_{R^k} := \{\vec{z} \in \mathcal{P}_k : \ R^k(\vec{z}) \geq R^k\}$ and $\mathcal{P}_k$ and is a tangent to the set $\mathcal{L}^k_{R^k}$. We consider a new sub-game for provider k where the constraint is $\mathcal{Q}^k = \{\vec{x} \in \mathcal{P}_k : \sum_i \gamma_i^k x_i \leq 1\}$ and the utility functions are linear, i.e. for some $v_i^k > 0$, the utility function is $v_i^k x$, $x \geq 0$, for every user i. In particular, we take

$$v_i^k = U_i'(x_i) + U_i''(x_i) x_i^k, \text{ for each } i.$$

We note that the following holds with proof deferred to Appendix 7.6.

Claim 3 $v_i^k > 0$ *for every i and k.*

From Theorem 2, we have that for any Nash-equilibrium allocation $\vec{u}^k$ of the new game,

$$\max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i^k z_i \leq \kappa \sum_i v_i^k u_i^k$$

where κ is equal to 1 and $1 + 2/\sqrt{3}$ under price taking and price anticipating users, respectively.

We note the following claim whose proof is deferred to Appendix 7.7.

Claim 4 $\sum_i v_i^k u_i^k \leq \sum_i U_i'(x_i) x_i^k$.

It follows that

$$\max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i^k z_i \leq \kappa \sum_i U_i'(x_i) x_i^k. \tag{25}$$

We next consider the whole system with m providers. Let $v_i = \min_k v_i^k$ and define $V_i(x) = a_i + v_i x$ where a_i is chosen so that $V_i(x)$ is a tangent to $U_i(x)$. Since $U_i(x)$ is a non-negative concave function, we have $U_i(x) \leq V_i(x)$, for $x \geq 0$. Hence

$$\max_{\vec{z} \in \sum_k \mathcal{P}_k} \sum_i U_i(z_i) \leq \max_{\vec{z} \in \sum_k \mathcal{P}_k} \sum_i V_i(z_i) = \sum_i a_i + \sum_k \max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i z_i. \tag{26}$$

We next note the following. First, from the definition of δ-utilities,

$$a_i \leq \delta U_i(x_i). \tag{27}$$

Second, by the definition of v_i, $v_i \leq v_i^k$, for every i, so that

$$\max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i z_i \leq \max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i^k z_i \leq \kappa \sum_i U_i'(x_i) {x_i}^k$$

where the last inequality is by (25). Therefore,

$$\begin{aligned} \sum_k \max_{\vec{z} \in \mathcal{P}_k} \sum_i v_i z_i \quad &\leq \quad \kappa \sum_k \sum_i U_i'(x_i) {x_i}^k = \kappa \sum_i U_i'(x_i) x_i \\ &\leq \quad \kappa \sum_i U_i(x_i) \end{aligned} \tag{28}$$

where the last inequality is because $U_i(x)$ is a non-negative concave function.

Finally, using (27) and (28) in (26), we have

$$\max_{\vec{z} \in \sum_k \mathcal{P}_k} \sum_i U_i(z_i) \leq \delta \sum_i U_i(x_i) + \kappa \sum_i U_i(x_i) = (\delta + \kappa) \sum_i U_i(x_i)$$

which establishes the asserted claims. ■

6 Discussion and Related Work

Price discrimination by a profit-maximizing monopoly is a well known economic concept (Tirole [34], Varian [36]) and is known that in some cases it can improve social welfare; see, for example, Varian [35] for a necessary condition for this to happen. An architecture for price discrimination in the Internet networks was proposed by Odlyzko [26], namely, the Paris-Metro scheme. Under this scheme, the network resources are partitioned over several classes, each offering an individual price, and resources within a class are shared equally among the competing network connections. [26] is an architecture proposal, leaving open how to set prices and other implementation aspects. In the present paper, we consider a mechanism that induces price discrimination.

Shenker et al [32] challenged the marginal cost pricing as a design goal arguing that (i) marginal cost pricing may be insufficient to recover the costs, (ii) marginal costs may be inaccessible, and (iii) other goals than optimizing social welfare may be of interest. Furthermore, they proposed an architecture where a network provider determines prices at network edges, not by a distributed algorithm run over individual resources of this provider. The marginal cost pricing underlies the mechanisms that aim at maximizing social welfare; instead, we consider the revenue maximization.

Proportional allocation has been considered for processor sharing under Generalized Processor Sharing by Parekh and Gallager [28], and for resource sharing of real-time, time-shared systems by Stoica et al [33]. It also underlies Kelly's mechanism [14] for sharing of network resources. The mechanism is used for systems of multiple infinitely divisible resources of finite capacities by applying the mechanism at each individual resource, e.g. [29, 14, 10]. This is in contrast to the weighted proportional allocation mechanism that can be applied to resources with more general resource constraints (specified by general polyhedrons), hence, allowing for more complex systems than single processors or single network links.

A large body of literature focuses on the social welfare under the proportional allocation mechanism. Kelly [14] showed that under price taking users, the proportional allocation yields full efficiency. This was followed by Johari and Tsitsiklis [10] who showed that under price anticipating users, the worst-case efficiency is $3/4$. This was extended to resources with more general polyhedron constraints by Nguyen and Tardos [24] showing that if for each user with a given utility function there are $k \geq 1$ users with identical utility functions and competing for the same set of resources, then the social welfare is at least $1-\frac{1}{4k}$ of the maximum social welfare; for $k=1$, this boils down to the result of [10]. They also considered revenue, which we discuss later in this section. The network resource allocation game considered in [10, 24] assumes that users submit individual bids to resources that they want to use; for example, individuals bids are submitted to links of a network connection. This is unlike to Kelly's mechanism [14] where a scalar bid is submitted for the set of links of a network connection. The network resource allocation game with users submitting scalar bids was considered by Hajek and Yang [7], showing that the worst-case efficiency can be made arbitrarily close to 0, and that a Nash equilibrium may even not exist. A mechanism with users submitting scalar bids was considered by Johari and Tsitsiklis [11] where users choose desired allocations based on (marginal cost) prices; under price taking users, it is showed that full efficiency is achieved while under price anticipating users and affine marginal costs, a tight worst case efficiency is $2/3$. Efficiency of scalar-parameterized mechanisms where users are restricted to one-dimensional strategy spaces was studied by Johari and Tsitsiklis [12] and Yang and Hajek [39]. In particular, [12] showed that for a class of smooth market-clearing mechanisms where price discrimination is not allowed, the proportional allocation of Kelly [14] maximizes the worst-case efficiency; if price discrimination is allowed, then full efficiency can be achieved by an adaptation of Vickery-Clarke-Groves mechanism (e.g. see [8] for a survey). The maximization of the worst-case efficiency by Kelly's mechanism was also found by Maheswaran and Basar [20] but over a class of mechanisms for which the allocation is proportional to a function of the submitted bid and the payment is a function of the submitted bid. Finally, Cole, Dodis, and Roughgarden [4] considered a routing game where the goal is to adjust prices at network links so that a Nash equilibrium allocation maximizes the social welfare. All the work discussed in this paragraph focuses on the social welfare objective while in our work we consider the revenue maximization.

We now discuss price competition under revenue-maximizing providers which was studied under various assumptions. A pricing game of parallel links was considered by Acemoglu and Ozdaglar [1], and Hayrapetyan, Tardos, and Wexler [9]. In this pricing game, there is a flow demand through a set of parallel links where subsets of links are owned by independent providers who set prices selfishly trying to maximize their revenues. The flow is split across links that offer minimal marginal cost, equal to a sum of the price and a congestion cost. For each link, the congestion cost is assumed to be a function of the total flow through this link. Both [1] and [9] allow for convex non-decreasing link cost functions; [1] considers a fixed amount of demand with a constant marginal utility, while [9] assumes elastic demand specified by a utility function $U(x)$ such that $U'(x)$ is decreasing and concave (more general than assuming constant marginal utility). [1] finds tight lower bounds on the efficiency equal to $2\sqrt{2}-2 \approx 0.8284$ and $5/6 \approx 0.8333$, respectively, under the assumption that link cost functions are equal to zero at zero flow and without this assumption; [9] finds that in the two respective cases, the efficiency is at least $8/25 = 0.3125$ and $20/93 = 0.2151$ and for some utility functions $U(x)$ such that $U'(x)$ is convex, the efficiency can be made arbitrarily small. The pricing game was extended by Acemoglu and Ozdaglar [2] to a system of parallel-serial links, where each path consists of one or more links owned by independent providers. Under same assumptions as in [1] and assuming that the link cost functions are equal to zero at zero flow, they established a tight lower bound on the efficiency equal to $1/2$, for any equilibrium at which all the demand has to be fullfiled and each provider plays a strict best response; it is, however, showed that if the link cost functions are allowed to be positive at zero flow, the worst-case efficiency can be arbitrarily small. While we also consider a system of competing providers, these price competition games differ to our system in several respects. First, we consider an auction mechanism, not a price setting. Second, we consider systems with multiple strategic users and allow for heterogeneous utility functions. Third, we allow for price discrimination across individual users. Finally, rather than assuming soft resource constraints specified by a function of the total flow through a link, we allow each provider to have rather general resource constraints, specified by polyhedrons.

While most of the previous work focused on social welfare, some also considered the revenue. Shakkottai et al [31] considered the *price of simplicity*, defined as the ratio of the revenues under a simple pricing scheme and a revenue-maximization pricing; specifically, they considered a flat-rate and a generalized Paris-Metro scheme. Under assumption that utility functions are identical up to a multiplicative constant, and are non-negative, non-decreasing, and concave, they showed that the price of simplicity of the flat-rate pricing is near to 1 for some utility functions, but it can be arbitrarily small for worst-case (truncated linear) utility functions (order $1/\log(n)$ for large number of users n); similar results were established for the generalized Paris-Metro scheme. Furthermore, Maheswaran and Basar [21] considered the revenue of the class of generalized proportional allocation mechanisms introduced in [20], showing that for users with identical utility functions, Kelly's mechanism maximizes the revenue, but there exist cases with non-identical utility functions, for which Kelly's mechanism is sub-optimal. Nguyen and Tardos [24] considered the revenue of the network resource allocation game [10], showing that the revenue is a constant factor of the maximum revenue for a class of utility functions and assuming that for each user there is a number of users with identical utility functions and competing for the same set of resources. Our work differs from [31, 21, 24] in that we consider an auction mechanism applied by providers whose goal is to selfishly maximize their respective revenues, and we allow for heterogeneous utility functions.

Optimal auction design was considered with respect to social welfare (Vickery-Clark-Grove; see Nisan [25] for a survey) and revenue (Myerson [23]). This line of work admits a framework of Bayesian games with incomplete information about users' valuations; see Hartline and Karlin [8] for a survey. Our work is different in the allocation mechanism that we consider; in accommodating infinitely divisible resources with general polyhedron constraints; and admitting the framework of games with complete information.

A large body of work was devoted to mechanism design where the goal is to design optimal auctions that are incentive-compatible (i.e. truthful or strategy-proof); for an incentive-compatible mechanism, it is a dominant strategy for users to report their true valuations. For example, in this context, Fiat et al [6] considered profit-maximizing auctions for selling multiple indivisible items with cancellations (the seller is allowed to cancel the auction in case the collected revenue does not meet some a priori criteria). Using a competitive analysis, they provide a lower bound of 2 for a profit competitive ratio that they consider, along with algorithms with a competitive ratio of 4. Our work is different from this line of work as we do not impose the incentive-compatibility requirement and is in this sense closer to that studied in, for example, [14, 10, 24, 7, 11, 11, 39, 4, 1, 9, 20, 21].

Our work relates to auctions for sponsored search (see [18] for a survey), referring to online advertising by Internet search engines by selling ad slots that appear along side with search results to search keywords, using a generalized second price auction (GSP). The ad slots are allocated to a set of top ranked advertisers with respect to the weighted bids. The weight for an advertiser is the expected click-through-rate for this advertiser; each advertiser pays a fixed price per click where the price is the lowest value sufficient to retain the advertiser's rank. Edelman, Ostrovsky, and Schwartz [5] showed that GSP is not incentive-compatible and showed an equivalence to a generalized English auction; similar game was also studied by Varian [37]. In a recent work, Varian [38] considered the auctioneer profit per cost and provided some empirical estimates. The weighted proportional allocation applies to sponsor search scenarios by interpreting the allocation vector as the click-through-rates over the ad slots.

Finally, market-based approach for resource allocation of computational resources such as web farms or data centers was considered in prior work. For instance, Liu, Squillante, and Wolf [19] proposed a network flow model for maximizing the profit of a system of servers under users with diverse service-level agreements. The resources are partitioned over service classes using generalized processor sharing per each server, and the service-level agreements are offered per service class. For each service class, the service-level agreement is specified by a quality-of-service metric derived from the queuing theory (e.g. waiting time distribution). It is assumed that the service provider earns a fixed revenue per each user request satisfying its service-level agreement; this corresponds to users with linear utility functions. The flow rate assignments across classes and servers and the service rates per class at each server are a solution to an optimization problem with separable concave objective function across service classes. The mechanism that we consider could be applied to data center scenarios.

7 Conclusion

Our results show that the weighted proportional allocation mechanism provides competitive revenue and social welfare under a competition of selfish users who aim at maximizing their payoffs and selfish providers who aim at maximizing their revenues. It is a simple mechanism that can be applied by providers offering resources with rather general constraints, described by polyhedrons.

An interesting direction for future work is the design of iterative algorithms for weighted proportional allocation; in particular, exploration-exploitation type of algorithms that do not require a prior information about users' utility functions.

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Appendix

7.1 Proof of Proposition 1

A function $f(\vec{x})$ whose values are real or $\pm\infty$ and whose domain is a subset $\mathcal{S}$ of $\mathbb{R}^n$ is convex if and only if its epigraph set

$$\mathrm{epi}(f) = \{(\vec{x},\mu) : \vec{x} \in \mathcal{S}, \mu \in \mathbb{R}, f(\vec{x}) \leq \mu\}$$

is convex in $\mathbb{R}^{n+1}$ [30]. We want to show that $-R(\vec{x})$ is a convex function, hence consider $\mathcal{S} = \mathbb{R}^n_+$ and $R(\vec{x}) \geq \mu$, for $\mu \in \mathbb{R}_+$. Let for each i, $v_i(x) := U_i'(x)x$, for $x \geq 0$. From (5), note that $R(\vec{x}) \geq \mu$ can be written as

$$\sum_i \frac{v_i(x_i)}{v_i(x_i)+\mu} \geq 1 \Leftrightarrow \sum_i \frac{\mu}{v_i(x_i)+\mu} \leq n-1.$$

We need to show that

$$\mathrm{epi}(-R) = \left\{ (\vec{x},\mu) : \vec{x} \in \mathbb{R}^n_+, \mu \in \mathbb{R}_+, \sum_i \frac{\mu}{v_i(x_i)+\mu} \leq n-1 \right\}$$

is a convex set in $\mathbb{R}^{n+1}_+$. This is the same as saying that for any $(\vec{y},\mu_y),(\vec{z},\mu_z) \in \mathrm{epi}(-R)$, we have $(\vec{x},\mu_x) \in \mathrm{epi}(-R)$, for all $(\vec{x},\mu_x) = \lambda(\vec{y},\mu_y) + (1-\lambda)(\vec{z},\mu_z)$, $\lambda \in [0,1]$.

To contradict, suppose $(\vec{y},\mu_y),(\vec{z},\mu_z) \in \mathrm{epi}(-R)$ and $(\vec{x},\mu_x) \notin \mathrm{epi}(-R)$, for some $\lambda \in [0,1]$. We then have

$$\sum_i \alpha \frac{\mu_y}{v_i(y_i)+\mu_y} + (1-\alpha)\frac{\mu_z}{v_i(z_i)+\mu_z} \leq n-1, \text{ for all } \alpha \in [0,1] \tag{29}$$

and

$$\sum_i \frac{\mu_x}{v_i(x_i)+\mu_x} > n-1.$$

Under the assumption of the proposition, for each i, $v_i(x)$ is a concave function, hence, we have $v_i(x_i) \geq \lambda v_i(y_i) + (1-\lambda)v_i(z_i)$. It follows

$$\mu_x \sum_i \frac{1}{\lambda v_i(y_i)+(1-\lambda)v_i(z_i)+\mu_x} > n-1.$$

Furthermore, by the convexity of the function $1/x$, it follows

$$\mu_x \sum_i \lambda \frac{1}{v_i(y_i)+\mu_y} + (1-\lambda)\frac{1}{v_i(z_i)+\mu_z} > n-1.$$

Let $\lambda = \alpha\mu_y/\mu_x$, and rewrite the last inequality as

$$\sum_i \alpha \frac{\mu_y}{v_i(y_i)+\mu_y} + (1-\alpha)\frac{\mu_z}{v_i(z_i)+\mu_z} > n-1$$

which contradicts (29), hence showing that $(\vec{x},\mu_x) \in \mathrm{epi}(-R)$.

The second statement of the proposition that (4) is a convex optimization problem follows immediately as we showed that $R(\vec{x})$ is a concave function which under (4) is maximized over a convex set $\mathcal{P}$.

7.2 Proof of Claim 2

The minimum of $f(y)$ is achieved at a point y at which $f'(y) = 0$, which is the same as

$$y^4 - 5y^2 + 6y - 2 = 0.$$

Now, note

$$\begin{aligned} y^4 - 5y^2 + 6y - 2 &= y^4 - 2y^2 + 1 - 3y^2 + 6y - 3 \\ &= (y^2-1)^2 - 3(y-1)^2 \\ &= (y-1)^2[(y+1)^2 - 3]. \end{aligned}$$

Hence,

$$(y-1)^2[(y+1)^2-3] = 0$$

which has the following solutions $-\sqrt{3}-1$, $\sqrt{3}-1$, 1. We are only interested in the solutions in $[0,1]$. Noting that $f(\sqrt{3}-1) = 2\sqrt{3}-2 < 1/2$ and $f(1) = 1/2$, the claim is established.

7.3 Proof of Proposition 2

Following the same steps as in the proof of Theorem 2, we have that the social welfare at the Nash equilibrium is at least

$$\frac{ky}{1-y} + (1-ky)$$

and the maximum social welfare is at most

$$\frac{1}{(1-y)^2}(ky(1-y)+1-ky)$$

for some $0 \leq y \leq 1/k$. It follows that the efficiency is at least

$$f_k(y) = (1-y)\left(\frac{2-y}{1-ky^2} - 1\right)$$

for some $0 \leq y \leq 1/k$. It remains only to establish

$$\inf_{y \in [0,1/k]} f_k(y) = 1 - \frac{1}{2k} + o(1/k).$$

This follows by noting that for a minimizer y, $f_k'(y) = 0$, which is equivalent to

$$y^4 - \frac{5}{k}y^2 + \frac{2}{k}\left(2 + \frac{1}{k}\right)y - \frac{2}{k^2} = 0.$$

Since $y \leq 1/k$, we neglect the term y^4 as it is of smaller order than other terms, which amounts to solving a quadratic equation whose solution in $[0, 1/k]$ is given by

$$y = \frac{1}{5}\left(2 + \frac{1}{k} - \sqrt{4 - \frac{6}{k} + \frac{1}{k^2}}\right).$$

It readily follows that $y = \frac{1}{2k} + o(1/k)$ and plugging into $f_k(y)$ yields the asserted claim.

7.4 Proof of Lemma 1

7.4.1 Item i

It suffices to consider truncated linear functions, i.e. for $\alpha > 0$ and $y > 0$, $U(x) = \min\{\alpha x, y\}$, $x \geq 0$, as linear functions are a special case with $y = \infty$. Clearly, we have $U(b) - U'(b)b = 0$, for any $b \geq 0$, hence $\delta = 0$.

7.4.2 Item ii

Consider the tangent to $U'(x)$ at the point $x = a$; see Figure 5. This tangent forms the triangle BDF. Note that the area L is less or equal to the area of the triangle BDF. The side DF of the triangle is of length $-2U''(a)a$. The side FB of the triangle is of length $2a$. Hence, the area of the triangle is equal to $-2U''(a)a^2$. Now, note that the area W is greater or equal to the area of the rectangle $ACEF$. The sides of this rectangle are of length $-U''(a)a$ and a. Hence, the area of the rectangle is $-U''(a)a^2$. It follows that $L/W \leq 2$.

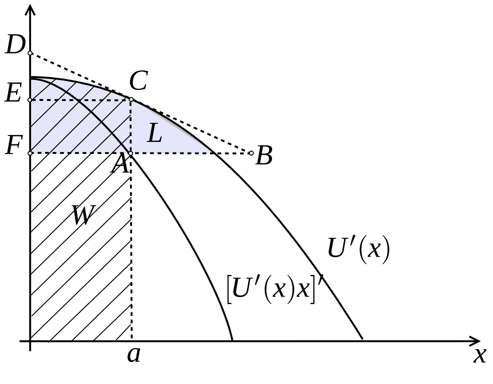

Figure 5: $U'(x)$ concave.

7.4.3 Item iii

We have

$$U'(x) = \frac{1}{c+x} \text{ and } [U'(x)x]' = \frac{c}{(c+x)^2}.$$

From $U'(b) = [U'(a)a]'$ we have

$$U'(b) = \frac{1}{c+b} = \frac{c}{(c+a)^2}$$

and

$$b = \frac{(c+x)^2}{c} - c.$$

It follows

$$\begin{aligned}\frac{U(b)-U'(b)b}{U(a)} &= \frac{2\log\left(\frac{c+a}{c}\right)+\left(\frac{c}{c+a}\right)^2-1}{\log\left(\frac{c+a}{c}\right)}\\ &= \frac{2\log(u)-u^2+1}{\log(u)} = 2-\frac{u^2-1}{\log(u)} := \varphi(u)\end{aligned}$$

where $u = c/(c+a)$. Since $(u^2-1)/\log(u) \geq 0$, we have $\varphi(u) \leq 2$, for all $u \in [0,1]$. This bound is tight; achieved at $u = 0$.

7.4.4 Item iv

We have

$$\begin{aligned}U(x) &= (c+x)^\alpha && (30)\\ U'(x) &= \alpha(c+x)^{\alpha-1} && (31)\\ [U'(x)x]' &= \alpha(c+x)^{\alpha-1}\left[1-(1-\alpha)\frac{x}{c+x}\right]. && (32)\end{aligned}$$

It follows

$$\frac{U(b)-U'(b)b}{U(a)} = (1-\alpha)\left[1-(1-\alpha)\frac{a}{c+a}\right]^{-\frac{\alpha}{1-\alpha}} + \alpha\frac{c}{c+a}\left[1-(1-\alpha)\frac{a}{c+a}\right]. \tag{33}$$

Remark Note that for $c = 0$, we have

$$\frac{U(b)-U'(b)b}{U(a)} = (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$$

which is independent of $a \geq 0$.

Let us consider the right-hand side with the following change of variables $u = a/(c+a)$,

$$f_\alpha(u) := (1-\alpha)\left[1-(1-\alpha)u\right]^{-\frac{\alpha}{1-\alpha}} + \alpha(1-u)\left[1-(1-\alpha)u\right].$$

It is not difficult to note that $f'_\alpha(u)$ is non-decreasing on $[0,1]$, hence $f_\alpha(u)$ is a convex function on $[0,1]$. It follows that the function $f_\alpha(u)$ over $u \in [0,1]$ achieves maximum at either $u = 0$ or $u = 1$, with values $f_\alpha(0) = 1$ and $f_\alpha(1) = (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$. We claim

$$\max_{u\in[0,1]} f_\alpha(u) = \begin{cases} (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}} & 0 \leq \alpha \leq \frac{1}{2}\\ 1 & \frac{1}{2} < \alpha < 1.\end{cases}$$

Indeed, $f_\alpha(0) \leq f_\alpha(1)$ if and only if $\alpha^\alpha \leq (1-\alpha)^{1-\alpha}$. The function x^x is non-decreasing, thus the last inequality holds if and only if $\alpha \leq 1-\alpha$, i.e. $\alpha \leq 1/2$. The claim follows.

Claim 5 $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}} \leq \frac{e}{2}$, *for all* $\alpha \in [0,1]$.

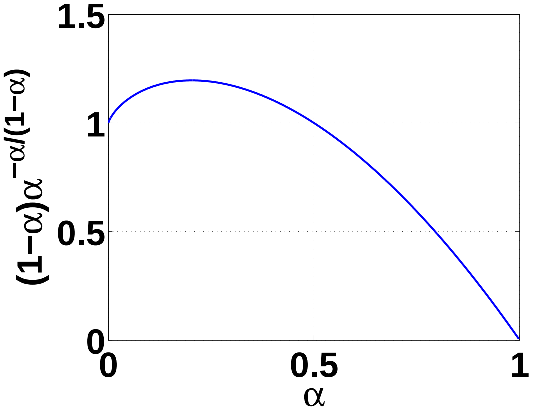

Figure 6: The function $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ versus α.

Remark The result establishes a uniform bound that holds for any $0 \leq \alpha < 1$, equal to $e/2 \approx 1.359$. This bound is not tight and can be improved, albeit slightly. Figure 6 shows the function $(1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$. Finding the maximum value numerically, we obtain the value 1.196.

Proof. Indeed, the function $f(\alpha) := (1-\alpha)\alpha^{-\frac{\alpha}{1-\alpha}}$ achieves the maximum value at the same points as the function $g(u) = \log f(\alpha)$. We have

$$g(\alpha) = \log(1-\alpha) - \frac{\alpha}{1-\alpha}\log(\alpha).$$

It is straightforward to obtain

$$g'(\alpha) = \frac{1}{1-\alpha}\left[2 + \frac{1}{1-\alpha}\log(\alpha)\right].$$

At a point α^* at which $g(\alpha)$ is maximum, we have $g'(\alpha^*) = 0$, which is equivalent to

$$\alpha^* = e^{-2(1-\alpha^*)}.$$

It follows

$$f(\alpha^*) = (1-\alpha^*)e^{2\alpha^*} = (1-\alpha^*)e^{-2(1-\alpha^*)}e^2 \leq e^2 \max_{x\in[0,1]} xe^{-2x} = \frac{e}{2}.$$

■

7.4.5 Item v

We have, for $\alpha \geq 0$,

$$U'(x) = \left(\frac{w}{c+x}\right)^{\alpha}.$$

It is not difficult to check that the function $U'(x)x$ is a concave function only if $0 \leq \alpha \leq 1$. Therefore, in the following we consider $\alpha \in [0,1]$.

Case 1: $\alpha \in [0,1)$. By straightforward calculus, we have

$$\frac{U(b) - U'(b)b}{U(a)} = \alpha\left[1 - \alpha\frac{a}{c+a}\right]^{-\frac{1-\alpha}{\alpha}} + (1-\alpha)\frac{c}{c+a}\left[1 - \alpha\frac{a}{c+a}\right].$$

Note that this is the same as (33) in Section 7.4.4, but α replaced with $1-\alpha$, hence the results in Section 7.4.4 apply by replacing α with $1-\alpha$. We obtain that $\delta = 1$ for $0 \leq \alpha \leq \frac{1}{2}$, and $\delta = \alpha(1-\alpha)^{-\frac{1-\alpha}{\alpha}}$, for $\frac{1}{2} < \alpha < 1$. From Claim 5, we can take $\delta = e/2$ for $\frac{1}{2} < \alpha < 1$.

Case 2: $\alpha = 1$. Consider the case $c = 0$. Note that $U'(b) = [U'(a)a]'$ is equivalent to

$$\frac{w}{b} = 0.$$

Thus, $b = \infty$, from which it follows

$$\frac{U(b) - U'(b)b}{U(a)} = \frac{\log(b) - 1}{\log(a)} = \infty, \text{ for any finite } a > 0.$$

This shows that $\delta = \infty$, and hence for $\alpha = 1$, $U(x)$ is not a δ-utility.

$\blacksquare$

7.4.6 Item vi

We have

$$U'(x) = \frac{\alpha^2}{\alpha^2 + x^2} \text{ and } [U'(x)x]' = \alpha^2 \frac{\alpha^2 - x^2}{(\alpha^2 + x^2)^2}.$$

Note that $[U'(x)x]' \geq 0$ if and only if $x \leq \alpha$, and $[U'(x)x]'$ is non-increasing in $[0, \alpha]$, hence, condition (i) of Definition 1 is verified with $x_0 = \alpha$.

From $U'(b) = [U'(a)a]'$ it follows

$$U'(b) = \alpha^2 \frac{\alpha^2 - a^2}{(\alpha^2 + a^2)^2} \text{ and } b = a\sqrt{\frac{3\alpha^2 + a^2}{\alpha^2 - a^2}}.$$

We need to show that

$$\varphi_\alpha(a) := \frac{U(b(a)) - U'(b(a))b(a)}{U(a)} \leq 2, \text{ for all } a \in [0, \alpha]$$

where

$$\varphi_\alpha(a) = \frac{\alpha \cdot \arctan\left(\frac{a}{\alpha}\sqrt{\frac{3\alpha^2 + a^2}{\alpha^2 - a^2}}\right) - a\alpha^2 \frac{\sqrt{(\alpha^2 - a^2)(3\alpha^2 + a^2)}}{(\alpha^2 + a^2)^2}}{\alpha \cdot \arctan\left(\frac{a}{\alpha}\right)}.$$

Note that $\varphi_\alpha(a) = \varphi_1(a/\alpha)$, hence it suffices to consider $\varphi_1(a)$ over $[0, 1]$.

Condition $\varphi_1(a) \leq 2$, for $a \in [0, 1]$, can be rewritten as

$$\arctan\left(a\sqrt{\frac{3 + a^2}{1 - a^2}}\right) - 2\arctan(a) \leq a\frac{\sqrt{(1 - a^2)(3 + a^2)}}{(1 + a^2)^2}, \; a \in [0, 1].$$

Clearly, the right-hand side is greater or equal to zero for all $a \in [0, 1]$. The claim follows by noting that the left-hand side is less than equal to zero for all $a \in [0, 1]$. To see this, note

$$\begin{aligned} g(a) \quad &:= \quad \arctan\left(a\sqrt{\frac{3 + a^2}{1 - a^2}}\right) - 2\arctan(a) \\ &= \quad \arctan\left(a\sqrt{\frac{3 + a^2}{1 - a^2}}\right) - \arctan\left(\frac{2a}{1 - a^2}\right) \end{aligned}$$

where equality follows from the elementary identity $\arctan(x) = 2\arctan\left(\frac{x}{1 + \sqrt{1 + x^2}}\right)$. Hence, $g(a) \leq 0$ if and only if

$$a\sqrt{\frac{3 + a^2}{1 - a^2}} \geq \frac{2a}{1 - a^2}$$

but this can be rewritten as $(1 - a^2)^2 \geq 0$, hence the proof. It can be easily checked that the equality in $\varphi_1(a) \leq 2$ is achieved for $a = 1$, hence $\delta = 2$ is tight.

7.5 Proof of Lemma 2

Items i and ii are straightforward to show. In the following, we show item iii.

Let $h = f + g$. Given $a \geq 0$, let $b \geq 0$ be such that

$$[h'(a)a]' = h'(b). \tag{34}$$

We need to show that

$$h(b) - h'(b)b \leq \delta h(a)$$

which corresponds to

$$f(b) - f'(b)b + g(b) - g'(b)b \leq \delta(f(a) + g(a)). \tag{35}$$

Let b_1 and b_2 be such that

$$[f'(a)a]' = f'(b_1) \tag{36}$$

$$[g'(a)a]' = g'(b_2) \tag{37}$$

and, without loss of generality, assume $b_1 \leq b_2$.

Since f and g are δ-utilities, the following two relations hold

$$\begin{aligned} f(b_1) - f'(b_1)b_1 &\leq \delta f(a) \\ g(b_2) - g'(b_2)b_2 &\leq \delta g(a). \end{aligned}$$

Hence,

$$f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2 \leq \delta(f(a) + g(a)). \tag{38}$$

In view of (35) and (38), it suffices to show that

$$f(b) - f'(b)b + g(b) - g'(b)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2. \tag{39}$$

Note that $[h'(a)a]' = [f'(a)a]' + [g'(a)a]'$. Combining with (34), (36), and (37), we observe

$$f'(b_1) + g'(b_2) = f'(b) + g'(b).$$

Using this identity it is not difficult to conclude that $b_1 \leq b \leq b_2$ and that we can rewrite (39) as

$$f(b) - f'(b_1)b + g(b) - g'(b_2)b \leq f(b_1) - f'(b_1)b_1 + g(b_2) - g'(b_2)b_2.$$

The latter inequality indeed holds if the following two inequalities hold

$$\begin{aligned} f(b) - f(b_1) &\leq f'(b_1)(b - b_1) \\ g(b_2) - g(b) &\geq g'(b_2)(b_2 - b) \end{aligned}$$

but the latter two inequalities are indeed true as $b_1 \leq b \leq b_2$ and both f and g are concave functions.

7.6 Proof of Claim 3

Taking partial derivative with respect to x_j^k on both sides of the equation (23), it is not difficult to establish that

$$\frac{U_j'(x_j) + U_j''(x_j)x_j^k}{(U_j'(x_j)x_j^k + R^k)^2} = \frac{\frac{\partial}{\partial x_j^k}R^k}{R^k} \sum_i \frac{U_i'(x_i)x_i^k}{(U_i'(x_i)x_i^k + R^k)^2}. \tag{40}$$

For an allocation $\vec{x}^k$ that maximizes R^k subject to the constraint $\sum_i \gamma_i^k x_i^k = 1$, we have either $x_i^k = 0$ or $(\partial/\partial x_i^k)R^k = \gamma_i^k \lambda$, for $\lambda > 0$. Therefore, either $x_i^k = 0$ or

$$\frac{U_i'(x_i) + U_i''(x_i)x_i^k}{(U_i'(x_i)x_i^k + R^k)^2} = \gamma_i^k \lambda > 0. \tag{41}$$

From this, if $x_i^k > 0$, we have $v_i^k > 0$. Otherwise, if $x_i^k = 0$, then $v_i^k = U_i'(x_i)$. From this and (24), it follows that $v_i^k > 0$ whenever $x_i > 0$. If $x_i = 0$, then $v_i^k = U_i'(0) > 0$ where the inequality holds as $U_i(x)$ is a non-constant, non-decreasing, and concave function.

7.7 Proof of Claim 4

From (41) and $\sum_i \gamma_i^k x_i^k = 1$, note

$$\sum_i \frac{U_i'(x_i) + U_i''(x_i)x_i^k}{(U_i'(x_i)x_i^k + R^k)^2} x_i^k = \left(\sum_i \gamma_i^k x_i^k\right)\lambda = \lambda.$$

Hence, (41) can be written as

$$\frac{U_i'(x_i) + U_i''(x_i)x_i^k}{(U_i'(x_i)x_i^k + R^k)^2} = \gamma_i^k \sum_j \frac{U_j'(x_j) + U_j''(x_j)x_j^k}{(U_j'(x_j)x_j^k + R^k)^2} x_j^k. \tag{42}$$

Now, let us use the following change of variables

$$y_i = \frac{U_i'(x_i)x_i^k}{U_i'(x_i)x_i^k + R_x^k} \text{ and } \tilde{y}_i = \frac{v_i^k u_i^k}{v_i^k u_i^k + R_u^k}$$

where R_x^k and R_u^k are the respective revenues for the game with original utility functions and the game with linear utility functions.

For the two games, the condition (23), respectively, corresponds to

$$\sum_i y_i = 1 \tag{43}$$

$$\sum_i \tilde{y}_i = 1. \tag{44}$$

Furthermore, the condition in (42) reads as

$$\begin{aligned}
\frac{v_i^k}{(U_i'(x_i)x_i^k + R_x^k)^2} &= \gamma_i^k \sum_j \frac{v_j^k x_j^k}{(U_j'(x_j)x_j^k + R_x^k)^2} \\
\frac{v_i^k}{(v_i^k u_i^k + R_u^k)^2} &= \gamma_i^k \sum_j \frac{v_j^k u_j^k}{(v_j^k u_j^k + R_u^k)^2}
\end{aligned}$$

which with our change of variables correspond to

$$(1 - y_i)^2 v_i^k = \gamma_i^k R_x^k \sum_j \frac{v_j^k}{U_j'(x_j)} y_j (1 - y_j) \tag{45}$$

$$(1 - \tilde{y}_i)^2 v_i^k = \gamma_i^k R_u^k \sum_j \tilde{y}_j (1 - \tilde{y}_j). \tag{46}$$

From the last two equations, we have

$$(1 - y_i)\sqrt{R_u^k \sum_j \tilde{y}_j (1 - \tilde{y}_j)} = (1 - \tilde{y}_i)\sqrt{R_x^k \sum_j \frac{v_j^k}{U_j'(x_j)} y_j (1 - y_j)}.$$

Summing over i and making use of (43) and (44), it follows that

$$R_u^k \sum_j \tilde{y}_j (1 - \tilde{y}_j) = R_x^k \sum_j \frac{v_j^k}{U_j'(x_j)} y_j (1 - y_j). \tag{47}$$

Combining with (45)-(46), we have

$$y_i = \tilde{y}_i, \text{ for every } i. \tag{48}$$

The assertion of the claim, with our change of variables, reads as

$$R_u^k \sum_j \frac{\tilde{y}_j}{1-\tilde{y}_j} \leq R_x^k \sum_j \frac{y_j}{1-y_j}$$

In view of (48), this is the same as

$$R_u^k \leq R_x^k$$

but this indeed follows from (47) as $y_i(1-y_i) = \tilde{y}_i(1-\tilde{y}_i) \geq 0$ and $v_i^k \leq U_i'(x_i)$ for every i.