Linear Algorithm for Shortest Path Through Select Orthants

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Let $V(\mathbb{R}^n)$ be the subspace of $\mathbb{R}^n$ containing the open all positive orthant, the open all negative orthant, and the closed orthants whose first i coordinates are non-positive and the remaining coordinates are non-negative for $1 \leq i \leq n$. We will show that there is a linear time algorithm for computing the Euclidean shortest path from a point in the all positive orthant to a point in the all negative orthant within $V(\mathbb{R}^n)$ for all $n \geq 2$. Thinking of the orthants missing from $V(\mathbb{R}^n)$ as obstacles, this is a specialization of the Euclidean shortest path with obstacles problem, which Canny and Reif [2] proved is NP-hard in $\mathbb{R}^3$. Our algorithm is also directly applicable to work in efficiently computing the shortest distance between two phylogenetic trees in the space of trees introduced by Billera et al. [1].

Let $P_i = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_j \leq 0 \text{ if } j < i;\ x_j = 0 \text{ if } j = i;\ x_j \geq 0 \text{ if } j > i\}$ for all $1 \leq i \leq n$. Then P_i is the boundary between the i-th and $(i+1)$-st n-dimensional orthants in $V(\mathbb{R}^n)$ (counting from the all positive orthant to the all negative orthant). Let $A = (a_1, ..., a_n)$ be the start point and let $B = (-b_1, ..., -b_n)$ be the target point, where $a_i, b_i > 0$ for all $1 \leq i \leq n$. We want to find the Euclidean shortest path between them in $V(\mathbb{R}^n)$, which is equivalent to finding the Euclidean shortest path from A to B that passes through $P_1, P_2, ..., P_n$ in that order, and shall be called the *shortest ordered path.* Furthermore, this is also a specialization of the touring convex bodies problem [4], for which a polynomial time algorithm in $\mathbb{R}^2$ was given in [3], since without loss of generality we can replace P_i with the intersection of P_i and the rectangular prism with opposing vertices A and B. From now on, we consider the problem as a touring one.

The following lemma establishes when the shortest ordered path from A to B is a straight line.

Lemma 1. *The line $\overline{AB}$ passes through the regions $P_1, P_2, ..., P_n$ in that order and has distance $\overline{AB} = \sqrt{\sum_{i=1}^{n}(a_i + b_i)^2}$ if and only if $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq ... \leq \frac{a_n}{b_n}$.*

This lemma can be proven by parameterizing the line $\overline{AB}$ with respect to time, and then finding the conditions such that the line intersects P_i no later than it intersects P_{i+1} for all $1 \leq i \leq n-1$.

In general, we do not have $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq ... \leq \frac{a_n}{b_n}$. However, we can keep reducing the problem to lower dimensional spaces until this is true. More specifically, we will define a locally shortest ordered path. If $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq ... \leq \frac{a_n}{b_n}$ does not hold, then all locally shortest ordered paths are in a subspace of $\mathbb{R}^n$ isometric to $\mathbb{R}^{n-1}$. Repeat this reasoning in the lower dimensional space until its corresponding ratios do form a non-descending sequence. We then show there is only one locally shortest ordered path in this space, which is thus the unique shortest ordered path.

To start, we define a locally shortest ordered path. An *ϵ-neighborhood* of a path is all points in $\mathbb{R}^n$ within $\epsilon > 0$ of at least one point on that path. An *locally shortest ordered path*, q, is a path from A to B which passes through $P_1, ..., P_n$ in that order, and for which there exists some $\epsilon > 0$ such that there is no shorter path q' from A to B contained in the ϵ-neighbourhood of q that also passes through $P_1, ..., P_n$ in that order. For all i, let p_i be the first point at which the locally shortest ordered path under consideration intersects P_i. Then each locally shortest ordered path is a straight line between p_i and p_{i+1} for all $1 \leq i \leq n-1$, only intersects with P_i at the point p_i for all i, and only bends where $p_i = p_{i+1}$ for some $1 \leq i \leq n-1$. Thus each coordinate of q decreases piecewise linearly, possibly decreasing at a different rate while positive than while negative.

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Lemma 2. *Consider a locally shortest ordered path q from A to B through P_1, ..., P_n. Let $\{M_j\}_{j=1}^m$ be any ordered partition of $[n]$ such that $i, l \in M_j$ implies $p_i = p_l$. Then this path exists in a subspace of $\mathbb{R}^n$ isometric to $\mathbb{R}^m$.*

If $p_i = p_{i+1}$, then the i-th and $(i+1)$-st coordinates of q change from positive to negative at the same point on q. Therefore, the rates at which each decreases to and increases from 0 are related, which implies that q lies in a subspace of $\mathbb{R}^n$ isometric to $\mathbb{R}^{n-1}$. We use this isometry to map q into $\mathbb{R}^{n-1}$, where q starts at $\tilde{A} = (a_1, a_2, ..., a_{i-1}, \sqrt{a_i^2 + a_{i+1}^2}, a_{i+2}, ..., a_n)$ and ends at $\tilde{B} = (-b_1, ..., -b_{i-1}, -\sqrt{b_i^2 + b_{i+1}^2}, -b_{i+2}, ..., -b_n)$.

We will apply the following constraint inductively to get a unique locally shortest ordered path.

Lemma 3. *Let q be any locally shortest ordered path from A to B. If $\frac{a_1}{b_1} \leq \frac{a_2}{b_2} \leq ... \leq \frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1}}$, then q travels through the intersection of P_i and P_{i+1}.*

We assume $p_i \neq p_{i+1}$ and prove this lemma by contradiction. The proof involves analyzing the times when q intersects each P_i in the case where q bends at p_{i+1} and in the case where it does not.

If we apply lemma 3 to find the least i for which $p_i = p_{i+1}$, we can then use lemma 2 to reduce the problem to finding the locally shortest ordered path in $\mathbb{R}^{n-1}$. Now apply lemma 3 to the corresponding ratio sequence in $\mathbb{R}^{n-1}$, and then lemma 2 if two consecutive ratios are still out of order. Keep applying these two lemmas until we have a non-descending ratio sequence $\frac{\tilde{a}_1}{\tilde{b}_1} \leq \frac{\tilde{a}_2}{\tilde{b}_2} \leq ... \leq \frac{\tilde{a}_m}{\tilde{b}_m}$ in $\mathbb{R}^m$, where $m < n$.

Theorem 4. *There is a unique locally shortest ordered path between $\tilde{A} = (\tilde{a}_1, ..., \tilde{a}_m)$ and $\tilde{B} = (-\tilde{b}_1, ..., -\tilde{b}_m)$ in $\mathbb{R}^m$, with distance $\sqrt{\sum_{i=1}^m (\tilde{a}_i + \tilde{b}_i)^2}$. This is the length of the shortest ordered path between A and B in $\mathbb{R}^n$.*

By lemma 1, we see that the line $\overline{\tilde{A}\tilde{B}}$ is the unique locally shortest ordered path between $\tilde{A}$ and $\tilde{B}$, and has distance $\sqrt{\sum_{i=1}^m (\tilde{a}_i + \tilde{b}_i)^2}$. Since we mapped from $\mathbb{R}^n$ to $\mathbb{R}^m$ by repeated isometries, both the length of the path and the order it passes through $P_1, ..., P_n$ (or their images) remain the same. Thus, there is only one locally shortest ordered path in $\mathbb{R}^n$ and it must be the globally shortest ordered path we are looking for. This is also the globally shortest path through $V(\mathbb{R}^n)$.

To explicitly calculate the shortest distance, consider the ratio sequence $\frac{a_1}{b_1}, \frac{a_2}{b_2}, ..., \frac{a_n}{b_n}$. For the first i such that $\frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1}}$ replace the i-th and $(i+1)$-st ratios with the ratio $\frac{\sqrt{a_i^2 + a_{i+1}^2}}{\sqrt{b_i^2 + b_{i+1}^2}}$. Repeat this step for the resulting ratio sequence starting at the $(i-1)$-st ratio, until we have a non-descending ratio sequence $\frac{\tilde{a}_1}{\tilde{b}_1} \leq \frac{\tilde{a}_2}{\tilde{b}_2} \leq ... \leq \frac{\tilde{a}_m}{\tilde{b}_m}$. The distance of the corresponding path is $\sqrt{\sum_{i=1}^m (\tilde{a}_i + \tilde{b}_i)^2}$.

Theorem 5. *The above algorithm has complexity $O(n)$.*

References

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