Partial Solutions to Homework 10 for MATH 336

28C # 2

Let $\log_\alpha \beta = r$. We are assuming that $\beta \neq 0$, and thus has an inverse, β^{-1}. Then

$$\begin{aligned}
\beta\beta^{-1} &\equiv 1 \pmod 7 \\
\log_\alpha(\beta\beta^{-1}) &\equiv \log_\alpha(1) \pmod 7 \\
\log_\alpha(\beta) + \log_\alpha(\beta^{-1}) &\equiv 0 \pmod 7 \\
r + \log_\alpha(\beta^{-1}) &\equiv 7 \pmod 7 \\
\log_\alpha(\beta^{-1}) &\equiv 7 - r \pmod 7
\end{aligned}$$

Therefore, we have shown that $\log_\alpha(\beta^{-1}) = 7 - r$.

By the table at the top of pg. 425, $\log_\alpha(\alpha + 1) = 3$. Then by what we just proved, and $\log_\alpha((\alpha+1)^{-1}) = 7 - 3 = 4$. This implies, by the table, that $(\alpha+1)^{-1} = \alpha^2 + \alpha$.

28C # 21

Let F be a field with 27 elements. F has characteristic 3, so by definition of characteristic, $1+1+1 = 0$. This means that $2^2 = 1$ and $-1 = 2$.

Let β be any element of F, such that $\beta \neq 0, 1, 2$. Since the orders of β and 2β must divide $\phi(27) = 26$, β and 2β can only have order 1, 2, 13, or 26. $\beta \neq 1$, so β doesn't have order 1. If 2β has order 1, then $2\beta = 1$. This implies that $\beta = 2$, which is a contradiction. If β has order 2, then $\beta^2 \equiv 1$. Since F and β satisfy the hypotheses of exercise E18 in 8A, we have that $\beta = 1$ or $\beta = -1 = 2$, which are both contradictions. Therefore, β must have order 13 or 26. If $(2\beta)^2 = 2^2\beta^2 = 1$, then $\beta^2 = 1$, which we just showed cannot be true. Therefore, 2β also has order 13 or 26.

If β has order 13, and hence is not a primitive element, then $(2\beta)^{13} = 2^{13}\beta^{13} = 2 \cdot 1 = 2$. Therefore, 2β does not have order 13, and thus must have order 26 and be a primitive element.

If β has order 26, and hence is a primitive element, then $(2\beta)^{13} = 2^{13}\beta^{13} = 2\beta^{13}$. But $(\beta^{13})^2 = \beta^{26} = 1$. By E18 in 8A, this implies that $\beta^{13} = 1$ or $\beta^{13} = -1 = 2$. But $\beta^{13} \neq 1$, since β has order 26. Therefore, $\beta^{13} = 2$, and hence $(2\beta)^{13} = 2\beta^{13} = 2 \cdot 2 = 1$. This implies that 2β has order 13, and thus is not a primitive element.

Therefore, we have shown that either β or 2β, but not both, is a primitive element.